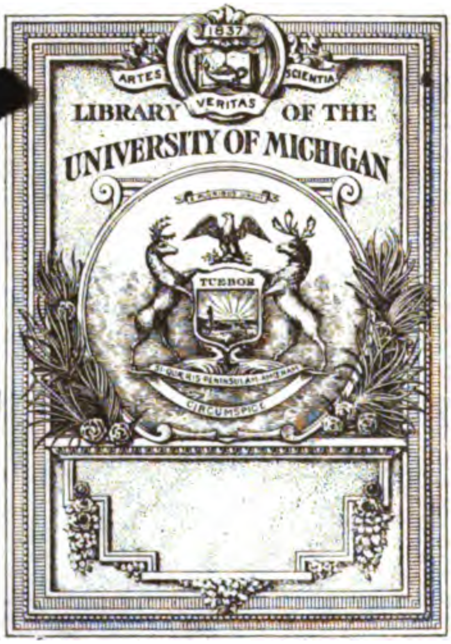
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THE ELEMENTS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS

BASED ON KURZGEFASSTES LEHRBUCH DER DIFFERENTIAL- UND INTEGRALRECHNUNG, VON W. NERNST, O. Ö. PROFESSOR DER PHYSIKAL. CHEMIE A. D. UNIV. GÖTTINGEN, UND A. SCHÖNFLIES, A. O. PROFESSOR DER MATHEMATIK A. D. UNIV. GÖTTINGEN

BY

J. W. A. YOUNG

ASSISTANT PROFESSOR OF MATHEMATICAL PEDAGOGY IN THE UNIVERSITY OF CHICAGO

AND

C. E. LINEBARGER

INSTRUCTOR IN CHEMISTRY AND PHYSICS IN THE LAKE VIEW HIGH SCHOOL, CHICAGO

NEW YORK

D. APPLETON AND COMPANY

1900

PREFACE

THE Differential and Integral Calculus is of the first importance, whether regarded from the standpoint of pure mathematics or from that of the natural sciences.

Without an elementary knowledge of this subject, even the names of the modern developments of pure mathematics are inexplicable jargon, while an introduction to the Calculus suffices to open to one the possibility of a vague apprehension of the problems of advanced research in mathematics, and of the results which the industry and acumen of the mathematicians of the day are continually producing, somewhat similar to that which he may have of the problems and results of chemistry, of Latin, of history, of biology. To the cultured man elliptic integrals, gamma functions, differential equations should be terms equally intelligible with the carbon compounds, the subjunctive mood in indirect discourse, feudalism, protoplasm. An elementary acquaintance with the Calculus makes this possible and brings mathematics within the range of the interest which one who leads an intellectual life feels in the varying phases and achievements of the intellectual activity of the day. A branch of study to which so large a share of the time of education is given up as is devoted to mathematics should surely not be dropped just when that subject is reached which in a few months would, in a sense, round out the attainments of all the previous years of work, and which offers a point of vantage for a general survey of the field of mathematics such as can nowhere earlier be found.

From the point of view of the natural sciences the subject is equally important. From its very discovery, the Calculus has been the indispensable handmaid of the physicist; for him its most complicated machinery has been put into motion; its most formidable engines have even been devised especially for him. But the chemist is now also calling upon the Calculus for aid. The day has come, "when a sign of differentiation or integration must cease to be an unintelligible hieroglyphic for the chemist, if he does not wish to run the risk of losing all comprehension of the development of theoretic chemistry. For it is a fruitless labor to squander page after page in a vain attempt to explain that which one equation makes perfectly clear to him who is initiated into the mysteries of the Calculus."* A similar day is approaching for all the natural sciences. In fact, the Calculus and the mathematical formulation of the phenomena of nature are inseparable.

With these ideas of the importance of the subject, one of the authors of the following work has for several years been giving courses of instruction in the elements of the Calculus, including a broad survey of its principles and methods, and a brief sketch of its ramifications throughout modern mathematics, but excluding the more complicated problems and the more difficult computations which should be taken up in the detailed treatment that is necessary as a basis for further mathematical study.

In consonance further with the view of Herbart, as quoted by Klein,† that mathematics is uninteresting to five-sixths of the students unless it is brought into direct connection with the applications; and of Clifford,‡ that "every connection

* Jahn, *Grundriss der Elektrochemie*, quoted in the preface to Nernst-Schönflies, *Differential- und Integralrechnung*.

† Klein, *Nichteuklidische Geometrie* (Lithographed lectures), I., p. 362.

‡ Clifford, *Common Sense of the Exact Sciences*, p. 257.

between two sciences is a help to both of them," applications of the Calculus were made, as occasion offered, to problems of the natural sciences as well as of pure mathematics; but the lack of a text-book written in the same spirit was felt to be a hindrance to the attainment of the best results.

Early in 1896, the work:

Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie, von W. Nernst, o. ö. Professor der Physikal. Chemie a. d. Universität Göttingen, und A. Schönflies, a. o. Professor der Mathematik a. d. Universität Göttingen,

which appeared in the latter part of 1895, was brought to his attention by his colleague in the present work. The latter, a pupil of Nernst, had been duly authorized to translate the German book into English, and had made some progress in doing so. While the German work was intended primarily for chemists, it appeared, even in this form, better suited for use as text in the courses in the Calculus just mentioned, than any available work in English; and it seemed possible to add to its efficiency for this purpose by alterations having as aim to enlarge the mathematical contents, to increase mathematical rigor, and to adapt the style of presentation to American methods of instruction. Accordingly the work of preparing in collaboration a translation, revised and adapted for use in American Colleges and Technical Schools, was undertaken by the present writers. It was thought that this could be accomplished by additions which could readily be indicated by some distinctive mark, leaving the original in the main intact. The actual work, however, gradually made it apparent that the alterations which seemed desirable were so serious that the German authors could no longer be held responsible for the matter in its new presentation. The alterations have been so numerous and so interwoven with the whole fabric, ranging from changes in phraseology to the adoption of a new

method, from the most trifling omissions and additions to the omission and addition of whole chapters, that though the present work is most closely based upon the valuable work of Professors Nernst and Schönflies, it is but just to the latter that the present writers bear the entire responsibility for the work as here presented. The last chapter, "The Differentiation and Integration of Functions found Empirically," is simply a translation of the corresponding chapter of the German text; otherwise only the present writers are to be held responsible for whatever defects may be found in the following work, while its merits are to be ascribed in a very large measure to the German work upon which it is based.

In the topics taken up, and in the extent of their treatment, the German work has served as model; a very great part of what appears (especially the presentation of those topics which have long since become the common basis of all elementary works on this subject) is a translation, more or less close, from the German work; the distinctive feature of the latter, *viz.* the continual use of illustrative examples from the natural sciences, is likewise a characteristic feature of the present work. With a few exceptions, the illustrations of this sort used in the German text have been retained, and a number of additional ones introduced. But here, and wherever necessary, the mode of presentation has been radically changed. The German authors, after using the method of limits to establish the fundamental rules, soon introduce the method of differentials and make very considerable use of it thereafter. Though this method may be satisfactory from the physico-chemical standpoint, it seemed that in a mathematical text-book the method of limits should be used exclusively.

The writers believe that the work as herewith presented is the first elementary American presentation of the Calculus

in which this is done. Even those writers who introduce the subject by the method of limits, usually take up the method of differentials sooner or later, and to a greater or less extent. We regard this as decidedly inadvisable. We believe that even with methods of equal rigor, the beginner is on the whole more confused than helped, if a subject is presented to him for the first time according to two or more different methods. In our subject, the methods do not stand on the same plane as to accuracy. If a logically sound "method of differentials" is set up, it is no longer a distinct *method*, but only a different *terminology*, and confusion is almost certain to result from the use interchangeably of two sets of names for one set of ideas.

The chief difficulty of the method of limits lies in acquiring a clear understanding of the notion of a limit, and of its application to functions of one or more variables. This notion cannot be eliminated from even elementary mathematics, and attempts to evade it must be futile. Experience has shown that when fairly faced, it offers no serious difficulty to beginners, and when once it has been grasped, the development of our subject proceeds with ease, security, and economy of energy. One who has once acquired a fair knowledge of the Calculus by the method of limits, will have no difficulty of consequence in understanding the differential notation, should he happen to meet it later in his reading.

A word as to *rigor*. The present may be styled the "Age of Rigor" in the development of the Calculus. The keen thinkers of the generation of mathematicians just passing away found much in the work of the older masters that needed more precise formulation and more strict treatment. But as it was not natural or easy in the beginning of the subject to perceive all the underlying subtle discriminations which were seen later, so now it is highly inadvisable, if not

quite impossible, to present the subject to beginners in the careful form which the modern notion of rigor demands. Nevertheless, an introduction to the Calculus to-day should profit by the results of the nineteenth century's labors. In the present work the fundamental principles and methods have been treated in as careful a manner as seemed consistent with the elementary character of the work, and throughout the aim has been to give a presentation in harmony, at least, with the more strict treatment, and permitting later extension upon the foundation here laid.

In the choice of the exercises, the aim has been to exemplify, to clarify, and to fix in mind the principles which have been explained. To this end the exercises are simple in character, so that the application of the principle may not be obscured by complexity of computations. The number of the exercises is thought to be sufficient to attain the end in view, though not sufficient to insure the attainment of great dexterity in the handling of long and intricate expressions. This skill can be attained only by considerable practice *after* the principles are understood. As the topics treated are those usually taken up in accordance with well-established usage, the teacher who desires to do so, can readily select supplementary exercises from other sources.

It is much more difficult to secure from works on the physical sciences good illustrative examples sufficiently simple to be available for the present work. When found, they usually require alteration in form to bring them into uniformity with one another and with our treatment of the subject. Accordingly the number of such examples included is perhaps larger than needful for any one class, permitting the teacher to make such selection as he may deem wise. The illustrations from the physical sciences are usually independent of one another, and any or all of them may be omitted without breaking the course of the mathematical development.

The chapters are also in the main independent of one another, at least to such an extent as to permit whatever omissions or variations in the order of reading are likely to be desired. In particular, it is possible, without serious inconvenience, to take up first all the chapters relating to the Differential Calculus.

The first chapter consists of an introduction to Analytic Geometry, and contains all that is presupposed from this subject in the remainder of the book. This chapter may be omitted by those who have already had a course in Analytic Geometry.

The historical notes are in some instances based on examination of the original sources by us, but usually on the authority of Cantor.*

It is hoped that the work as here presented may be helpful to students of several types: —

To the *student of mathematics* as a *pre-view*. In perhaps every branch of mathematics the subject-matter may readily be divided roughly into fundamental principles, methods, and results, which are not difficult of comprehension, and their combinations and generalizations, which may grow to any degree of complexity. It is usually a mistake to combine these two divisions of the subject-matter in a first presentation. When once the fundamental principles and results are well in hand, the attention can be given entirely to the steps by which these are combined into more elaborate results, and thus, taken in due order, the complex consequences offer no more difficulty than the simple elements; while detailed treatment of topics whose fundamental principles have not been thoroughly digested, entails unnecessary difficulty if not absolute failure. To the prospective student of mathematics the present work offers such a first general

* Cantor, *Vorlesungen über Geschichte der Mathematik*, Bde. II. III.

survey of the field of the Calculus, and, if desired, of Analytic Geometry also.

To the *general student* as a part of liberal culture. The reasons for believing that a course in the Calculus should round out the mathematical study of the general student have already been touched upon. The earliest stage at which work in mathematics may properly cease in the attainment of a liberal education has been well indicated by Hill.: * —

"How far is mathematical study, then, to be insisted upon as necessary to a 'liberal' education? Certainly no education can be called 'liberal' which has not enabled the recipient of it to perceive the mathematical necessity that runs through all natural relations, and to make those calculations which are needed in the exact sciences."

To the *student of natural science*, as giving sufficient of an acquaintance with the Calculus to render certain important recent developments in his domain intelligible.

To the *student of astronomy, of advanced physics, of technology*, for the same reasons as to the student of mathematics.

Various professional colleagues were good enough to look over portions of the proofs and to give us valuable suggestions which we have utilized. We wish to thank all of these gentlemen most heartily for this assistance, as well as the publishers for facilitating the work in every way in their power.

J. W. A. YOUNG,
C. E. LINEBARGER.

* Hill, *The American College in Relation to Liberal Education.* Inaugural address as President of the University of Rochester, p. 19.

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CALCULUS

CHAPTER I

THE ELEMENTS OF ANALYTIC GEOMETRY

ART. 1. **Graphic representation.** During the last few decades, the Graphic Methods have developed more and more into general and useful aids in investigation. They are employed to great advantage in the physical as well as in the descriptive sciences, such as Geography, Meteorology, Physiology, Sociology, etc.; and are applicable, in short, wherever laws and rules are considered in connection with numbers. The peculiar value which these methods possess lies in the substitution of geometric figures for numerical tables, the relations of the numbers being thus made directly apparent to the eye. A few examples will suffice to show the importance and applicability of the graphic method.

I. As a first illustration we reproduce* a table and diagram giving the value of various elements of growth of the United States at the times indicated. Much that must be slowly gleaned from the table of statistics alone is told by the diagram at a glance. From it we can answer at once such questions as: When were the carrying trades in Ameri-

* W. J. McGee, *The National Geographic Magazine*, September, 1898.

can and foreign bottoms equal? When did the density of population increase? When decrease? etc., etc.

II. What is known as Boyle's Law states the relation in which the pressure and the volume of a gas stand when all

Elements of Growth	1790	1800	1810	1820	1830
Area square miles	827,844	827,844	1,999,775	1,999,775	2,059,043
Total population	3,929,214	5,308,483	7,239,881	9,633,822	12,866,020
Population-density	4.75	6.41	3.62	4.82	6.25
Wealth	—	—	—	—	—
Wealth per capita	—	—	—	—	—
Railway mileage	—	—	—	—	23
Carrying trade, foreign bottoms				$14,358,235	$14,447,970
Carrying trade, American bottoms				$118,201,462	$129,918,458

Elements of Growth	1840	1850	1860	1870
Area square miles	2,059,043	2,980,959	3,025,600	3,556,600
Total population	17,069,453	23,191,876	31,443,321	38,558,371
Population-density	8.29	7.78	10.39	10.84
Wealth	—	$7,136,000,000	$16,160,000,000	$30,069,000,000
Wealth per capita	—	$308	$514	$780
Railway mileage	2,818	9,021	30,626	52,922
Carrying trade, foreign bottoms	$40,802,856	$90,764,954	$255,040,793	$638,927,488
Carrying trade, American bottoms	$198,424,609	$239,272,084	$507,247,757	$352,969,401

Elements of Growth	1880	1890	1898 *a*	1898 *b*
Area square miles	3,556,600	3,556,600	3,556,600	3,681,236
Total population	50,155,783	62,622,250	71,000,000	79,000,000
Population-density	14.10	17.61	20.00	21.46
Wealth	$43,642,000,000	$65,037,091,197	—	—
Wealth per capita	$870	$1,036	—	—
Railway mileage	93,296	166,691	190,000	—
Carrying trade, foreign bottoms	$1,224,265.434	$1,371,116,744	$1,600,000,000	—
Carrying trade, American bottoms	$258,346,577	$202,451,086	$190,000,000	—

other properties are kept constant. A certain mass of gas is at one time under the pressure p, and, at another time, under the pressure p_1; if v and v_1 are the volumes cor-

responding to these pressures, then Boyle's Law states that v and v_1 are inversely proportional to p and p_1; that is, we have the proportion,

$$v : v_1 = p_1 : p,$$

or the equation,

(1) $$pv = p_1 v_1.$$

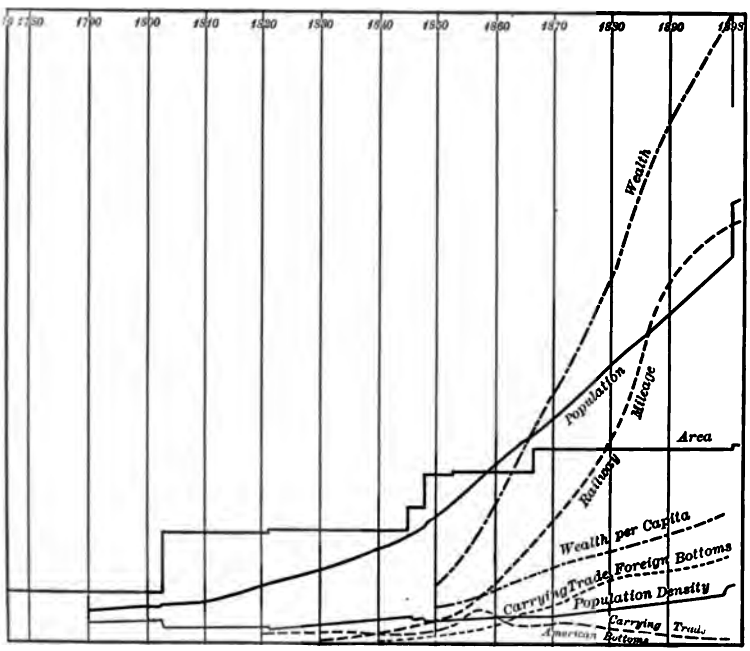

FIG. 1.

The graphic representation of Boyle's Law is obtained by making use of this equation in the following way. First of all, a series of corresponding values of pressure and volume is determined. If it be assumed that $p_1 = 1$ and the corresponding $v_1 = 1$, then equation (1) becomes $pv = 1$, and the

following table gives the corresponding values of p and v. If p equals,

0.1 0.2 0.5 1 2 4 5 etc.

the corresponding values of v are, respectively,

10 5 2 1 0.5 0.25 0.2 etc.

We now draw (Fig. 2) any horizontal line whatever and measure off on it from a point such as O, distances equal to the values of p respectively, that is, equal to 0.1, 0.2, 0.5, 1, 2, 4, etc.; at these points of division we erect perpendiculars whose lengths shall be equal to the corresponding values of v, or to 10, 5, 2, 1, 0.5, 0.25, etc. If the extremities of these perpendiculars be connected by a curved line, the curve thus plotted is the graphic representation of Boyle's Law. It is at once apparent that it is necessary to determine the position of quite a large number of points in order to obtain the course of the curve.*

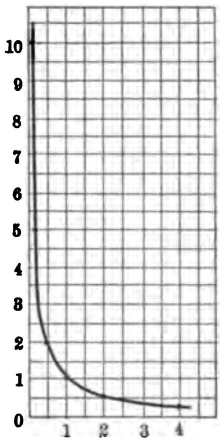

Fig. 2.

III. The air in a soap-bubble is compressed more than the external air, as is evident from its diminishing in size when the stem of the pipe used to blow it is left open. The pressure of the confined air and the diameter of the bubbles are given in the following table:

---

* It is convenient to employ the so-called "coördinate or cross-section paper," (*i.e.* paper divided into small squares by lines ruled at right angles,) for drawing the graphic representation of a law.

d Diameter of Bubble in Millimeters	p Pressure of Internal Air*	dp
7.55	3.00	22.65
10.37	2.17	22.50
10.55	2.13	22.47
23.35	0.98	22.88
27.58	0.83	22.89
46.60	0.48	22.37
		Mean = 22.63

An inspection of the columns of numbers shows that as d increases, p decreases, and that their product is nearly constant. Accordingly we may put $dp = constant$; the diameter of a soap-bubble is inversely proportional to the pressure of the air contained in it. Constructing the graphic representation, we obtain a curve similar in shape to that for Boyle's Law as given above.

IV. If a solid be brought into contact with a liquid, it often happens that the solid dissolves in the liquid, forming a solution. There is a limit to the amount of a solid that will dissolve, and when this limit is reached the solution is said to be **saturated** with the solid; the percentage by weight of the solid contained in the saturated solution is termed the **solubility**† of the solid in that liquid.

Solubility generally changes with a rise or fall of temperature, being different at different temperatures. The solubility of cane sugar in water has been determined to be:

* The unit for the pressure of the internal air is the pressure of air necessary to support or counterbalance a vertical column of water one millimeter in height.

† Of course, solubility may be expressed in other ways, as parts of liquid required to dissolve one part of solid, etc.

Temperature	0°	10°	20°	30°	40°	50°
Percentage of sugar dissolved . . .	65.0	65.6	67.0	69.8	75.8	82.7

If on any horizontal line we measure off distances equal to 0, 10, 20, 30, 40, 50 units on any chosen scale, at the points of division erect perpendiculars equal to 65.0, 65.6, 67.0, 69.8, 75.8, 82.7 units (on a scale which may or may not be the same as that employed in marking the temperatures), and then connect with straight lines the ends of the perpendiculars, we find we have constructed a broken line which resembles a curve. If the solubility had been determined at any intermediate temperatures, we should have a still more frequently broken line, and it is easily seen that the more values we determine for the solubility at different temperatures, the more closely does our broken line approach to a continuously curved line. When a sufficient number of pairs of values have been determined, we sketch a continuous curve passing through all the points. Such a curve is termed the **solubility curve** of sugar in water.

The practical value of such curves may be illustrated by the following considerations: Suppose we wish to know the solubility of sugar at 45°, a temperature not given in the table. First we sketch the curve through the points determined by the values of the table; then, at the distance 45 on the horizontal line, we erect a perpendicular touching the curve; the length of this perpendicular is the solubility sought. Moreover, the curve shows clearly, and at a glance, the general effect of temperature upon the solubility of sugar. At the lower temperatures the solubility remains nearly constant, but at the higher temperatures it increases more and more rapidly.

The method of sketching curves just described is rather

tedious and roundabout when an accurate figure is desired, since a large number of pairs of corresponding values are required. Unless the figure is accurately constructed, it is more likely to be misleading than to be of value. There is, however, a second method, which is much simpler, and is derived from the results of Analytic Geometry. When an equation is known which connects the corresponding numerical values of two related quantities as, for example, the pressure and volume of a gas, or the pressure and diameter of a bubble, *Analytic Geometry teaches at once and with complete generality the properties of the graphic representation of the relationship, which could be learned empirically only through a tedious operation.* This is true of every scientific law which can be formulated as an equation between two related quantities.

ART. 2. **Coördinates.** Analytic Geometry is based upon the same fundamental idea as the method of graphic representation, *viz.* the artifice of representing pairs of numbers geometrically by means of points.*

We draw in a plane two straight lines of indefinite length (Fig. 3), which may make any angle whatever with each other. Their point of intersection is designated by O, and the lines themselves by $X'OX$ and $Y'OY$. We take in the plane of the figure any point, as P, and draw through

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* Réné Descartes (1596–1650) was the first to make this artifice the basis of a systematic method of treating geometric problems. Under the simple title of *Géométrie*, Descartes (Lat. *Cartesius*) published in 1637 a little volume which was destined to introduce a new epoch in the study of geometry. Analytic Geometry, when treated according to the methods of Descartes, is frequently styled *Cartesian Geometry.* Descartes was a philosopher as well as a mathematician, traveled much, and led a varied and eventful life.

it lines parallel to the straight lines OX and OY; these lines cut off the distances OQ and OR, which, in this case, we suppose equal to 7 and 5 units, respectively. We term the distance OQ the **abscissa** of the point P, and the distance OR its **ordinate**, and usually denote these distances by x and y, respectively; more briefly, we say that for the point P,

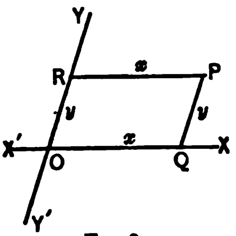

FIG. 3.

$$x = 7 \text{ and } y = 5.$$

When it is not necessary to distinguish the abscissa and ordinate, we call them jointly the **coördinates** of the point P.

A similar construction can be made for any other point in the plane. We thus obtain for every point a definite abscissa and a definite ordinate, or, as we may also say, a definite pair of coördinates expressed in numbers, such as 7 and 5. Conversely, if we wish to locate the point P' (Fig. 4), which corresponds to the numbers 2.5 and 3.5, we have to measure off on the straight line OX a distance

$$OQ' = 2.5,$$

and on OY a distance

$$OR' = 3.5,$$

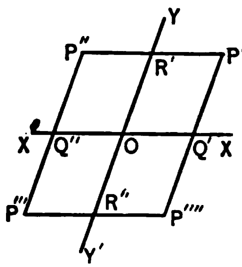

FIG. 4.

and draw parallel lines through the points Q' and R'; the point of intersection of these parallels is P'.

Since we can lay off distances on the lines XX' on either side of the point O, and likewise on YY', it might appear that we obtain not one, but four points, as P', P'', P''',

P''''. We avoid this, however, by establishing the rule that *such coördinates as are measured in opposite directions from O shall be given opposite signs.* Regarding, as heretofore, OQ' and OR' as positive, we must regard OQ'' and OR'' as representing negative coördinates; the points P', P'', P''', P'''' thus represent four different pairs of numbers, *viz.*,

$$+2.5, +3.5; \quad -2.5, +3.5; \quad -2.5, -3.5; \quad +2.5, -3.5.$$

The relationship between the points of the plane and the pairs of numbers is accordingly such that *a pair of numbers corresponds to every point of the plane, and vice versa, a point of the plane corresponds to every pair of numbers.** It is therefore customary to speak of a pair of numbers as a "point." Thus we say, "the point 2, 5," rather than "the point whose abscissa is 2 and whose ordinate is 5."

The two original straight lines are termed the **axes of coördinates**, the line $X'OX$ being the **axis of abscissæ**, and $Y'OY$ the **axis of ordinates.** Each has a positive and a negative half. They divide the plane into four parts, which are called **quadrants**, and are numbered as in trigonometry,

$$\begin{array}{c|c} \text{II} & \text{I} \\ \hline \text{III} & \text{IV} \end{array}$$

The point O, the point of intersection of the axes of coördinates, is called the **origin**, and the angle YOX the **coördinate angle**; if it be a right angle, as is found to be most convenient in practice, the coördinates are called **rectangular coördinates.**

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* The use of latitude and longitude to determine the location of a place upon the earth's surface, or on a map, is based upon the same idea, as is also (in a rude way) the specification of a house by its street and number.

Inasmuch as the coördinates of the point P (Fig. 3) are simply the numbers which give the length of OQ and OR, it follows that the distances PR and PQ are equal respectively to the abscissa and the ordinate of P. It is, accordingly, sufficient to draw one of the parallel lines from P in order to get the coördinates of P.

It is customary to give first the value of the distances parallel to the axis of abscissæ, as we have done above. The coördinates of the point are usually inclosed in parentheses, as "the point (4, 6)," and the axes are often called the **x-axis** and the **y-axis**, respectively.

It is easily seen that the axis of abscissæ contains all the points whose *ordinates are equal to zero;* also that the axis of ordinates is the *locus of all points having abscissæ equal to zero.* Finally, the origin is that point whose *coördinates are both equal to zero*, corresponding to the pair of numbers (0, 0).

When we represent graphically a point whose coördinates are given, we are said to **construct** or to **plot** the point. Similarly, any polygon or curve may be plotted when we know the coördinates of enough of its points to determine it completely. *Unless otherwise specified, the coördinate system is hereafter always supposed to be rectangular.*

EXERCISES I

1. Plot the following points:

(i.) $(2, 3)$;	(vi.) $(-\frac{3}{2}, -\frac{7}{4})$;	(xi.) $(-2, 0)$;
(ii.) $(4, 6)$;	(vii.) $(5, -9)$;	(xii.) $(-\frac{1}{2}, \frac{1}{2})$;
(iii.) $(-1, 7)$;	(viii.) $(0, -2)$;	(xiii.) $(-3, 0)$;
(iv.) $(2, -3)$;	(ix.) $(0, 3)$;	(xiv.) $(2, -7)$.
(v.) $(5, -35)$;	(x.) $(0, 0)$;	

2. Plot the straight lines passing through the following pairs of points:

(i.) (3, 0), (2, 3);
(ii.) (– 2, 6), (– 5, – 4);
(iii.) (– 3, – 2), (3, 2);
(iv.) (0, 0), (2, – 1);
(v.) (– 2, 5), (3, 5);
(vi.) (– 1, – 1), (1, – 1);
(vii.) (2, 2), (1, 1);
(viii.) (0, 0), (– 3, 0);
(ix.) (3, – 2), (3, – 5);
(x.) (– 1, – 2), (3, – 2);
(xi.) (2, – 2), (– 2, 2).

3. Plot the quadrilaterals whose vertices are the following sets of points:

(i.) (2, 1), (5, 2), (6, 4), (1, 5);
(ii.) (3, – 5), (– 4, – 2), (0, 0), (2, 3);
(iii.) (0, 4), (5, 3), (3, 0), (0, 0);
(iv.) (2, – 1), (5, – 1), (5, – 2), (2, – 2);
(v.) (5, 0), (5, 4), (– 5, 4), (– 5, 0);
(vi.) (1, 1), (2, 4), (– 1, 2), (– 2, – 1);
(vii.) (2, – 1), (5, – 4), (3, – 6), (0, – 3).

4. By inspection of the coördinates, tell in what quadrants the triangles lie (wholly or in part) whose vertices are the following sets of points:

(i.) (1, 2), (3, 1), (4, $\frac{1}{2}$);
(ii.) (– 1, 5), (– 2, 1), (– 4, 3);
(iii.) (2, – 1), (2, – 2), (3, 0);
(iv.) (1, 1), (– 1, 2), (– 2, 1);
(v.) (– 1, – 2), (– 4, – 1), (–1, – 1);
(vi.) (– 1, 3), (– 2, 2), (– 4, – 1).

5. (i.) What is the ordinate of any point which lies on a straight line parallel to the x-axis and at the distance 4 above it?

(ii.) What is the abscissa of any point which lies on a straight line parallel to the y-axis and at the distance d to the left of it?

(iii.) What is the abscissa of any point in the straight line perpendicular to the x-axis and intersecting it at the distance c from the origin?

(iv.) What relation exists between the ordinate and the abscissa of any point on the straight line which passes through the origin and bisects (a) the first and the third quadrant? (b) the second and the fourth quadrant?

ART. 3. **The fundamental principle of Analytic Geometry.** The result of exercise 5, iv. (*a*), immediately preceding, may be stated thus: *Throughout* the line AB (Fig. 5), $x = y$, and for all points not in AB, x is unequal to y (for x and y are the distances of any point from the axes, and all points without the bisector are unequally distant, *i.e.* for all such points $x \neq y$). That is, the relation $x = y$ is characteristic of this straight line and no other, and $x = y$ may hence be called *the* equation of this line.

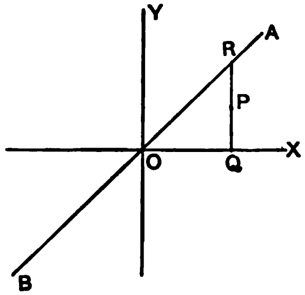

FIG. 5.

The method we have employed in this simple case holds generally: if any curve is given, and if we can succeed in finding a relation between x and y which holds for the coördinates of every point of the curve, and which does not hold for even a single point outside of the curve, the relation so found is characteristic of the curve, and if the relation can be expressed by an equation, the latter may be called **the equation of the curve.**

Conversely, if there is given an equation expressing a relation between the coördinates (x and y) of a point, the question arises: does there exist a curve such that the given relation exists between the coördinates of every point of it, and does not exist between the coördinates of any other point whatsoever? This question can usually be answered in the affirmative, as we proceed to illustrate in a few simple examples.

Suppose that we have given an equation between x and y, which we assume to be of the simplest possible form, as

(1) $$x + y = 4.$$

In the equations of elementary mathematics, the problem is to determine the values of the "unknown quantities," which will satisfy the equation; each problem usually has but a finite number of definite solutions. Now, however, the state of affairs is quite different; for there are countless pairs of numbers which, when introduced into our equation for x and y, will satisfy the equation. Such pairs of numbers are, for example:

$x=5$	$x=4$	$x=3$	$x=2$
$y=-1$	$y=0$	$y=1$	$y=2$
$x=1$	$x=-1$	$x=-1.5$	$x=-2$
$y=3$	$y=5$	$y=5.5$	$y=6$

Whatever number we may take for x, we can always obtain from equation (1) a corresponding value for y such that the pair of values so determined will satisfy the equation. For each of these pairs of numbers, a point of the plane can be determined whose coördinates are the numbers taken; in this way we obtain (Fig. 6) a boundless number of points, all of which lie upon a definite geometric curve that in this case seems, in the figure, to be a *straight line;* and it will be shown further on that this is actually the case.

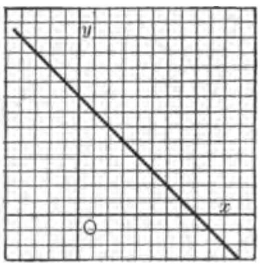

FIG. 6.

As a second example we take the equation,

$$(2) \qquad y^2 = 2x;$$

by assigning to x the values 0, 1, 2, 3 . . ., we obtain the following pairs of numbers which satisfy the equation:

$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	$y=\pm\sqrt{2}=\pm 1.4\ldots$	$y=\pm 2$	$y=\pm\sqrt{6}=\pm 2.45\ldots$
$x=4$		$x=5$	
$y=\pm\sqrt{8}=\pm 2.8\ldots$		$y=\pm\sqrt{10}=\pm 3.2\ldots$	

For every value of x there are two different values of y; thus for

$$x = 2,$$

$$y = +2,$$

and

$$y = -2,$$

so that both the point P' corresponding to the pair of numbers

$$x = 2$$

and

$$y = 2,$$

and the point P'' to which the numbers

$$x = 2$$

and

$$y = -2$$

belong, have coördinates which satisfy equation (2). Any number of such points can be found, and all of them lie upon a definite geometric curve (Fig. 7), which is called (Art. 5) a *parabola*.

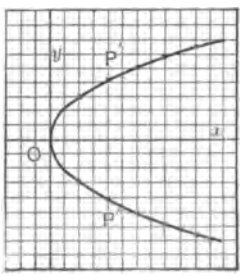

FIG. 7.

Similar considerations can be applied to every other equation between x and y. The countless pairs of numbers that can satisfy any equation always correspond to countless points, all usually lying upon a certain curve. The relation between the equation

and the curve may be expressed by stating that all points whose coördinates satisfy the equation in question lie upon the curve and (as will appear more fully later) the coördinates of every point upon the curve, used as values of x and y, will satisfy the equation. We often express this relation still more briefly by saying: *the given equation is the equation of the curve.* The curve corresponding to an equation is often called its **graph**, and also its **locus**. The curve regarded as the graph of an equation, may be considered the picture of the equation. Properties of the graph may be discovered by studying the equation, and *vice versa.* To do this is the fundamental purpose of Analytic Geometry.

A wide perspective now opens up before us; we see what must first be done in Analytic Geometry. Two problems present themselves at once: (1) to find the curve belonging to any given equation, and (2) to ascertain what is the equation of any given curve. It will be sufficient for our purposes to consider only the simplest cases of both these problems. Those of the second kind which we shall need will be treated in the sequel. We add a set of exercises containing a few simple examples of the first kind.

EXERCISES II

Construct the graphs of the following equations:

1. $x = 2y$.

2. $x + y = 0$.

3. $2x - y = 0$.

4. $y = x + 2$.

5. $x = y^2$.

6. $x^2 = 4y$.

7. $y + 2x + 4 = 0$.

8. $x^2 + y^2 = 36$.

9. $4x^2 = 9y^2$.

10. $x - y - 2 = 0$.

11. $9x - 4 = 2y$.

12. $x^2 + y^2 = 25$.

13. $x^2 = 5y - 2$.

14. $y = 6$.

ART. 4. **The equation of the circle.** *Given a circle of radius r; to find its equation.*

Let us take the axes of coördinates so that they shall intersect at the center of the circle and form a right angle (Fig. 8). The coördinates of any point of the circle as P_1 are OQ_1 and P_1Q_1, which we call x_1 and y_1. In the right triangle OP_1Q_1,

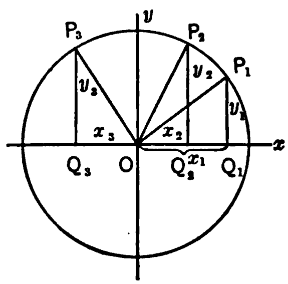

FIG. 8.

$$\overline{OQ_1}^2 + \overline{P_1Q_1}^2 = \overline{OP_1}^2;$$

or, if x_1, y_1, r, be substituted for OQ_1, P_1Q_1, and OP_1,

$$(1) \qquad x_1^2 + y_1^2 = r^2.$$

If P_2 be a second point of the circle, and OQ_2 and P_2Q_2, or, x_2 and y_2 be its coördinates, in a perfectly analogous manner, we obtain from the right triangle OP_2Q_2

$$\overline{OQ_2}^2 + \overline{Q_2P_2}^2 = \overline{OP_2}^2,$$

or (2) $$x_2^2 + y_2^2 = r^2.$$

In the same way it follows that, for any third point as P_3 with the coördinates OQ_3 and P_3Q_3, or x_3* and y_3,

$$\overline{OQ_3}^2 + \overline{P_3Q_3}^2 = \overline{OP_3}^2,$$

or (3) $$x_3^2 + y_3^2 = r_3^2.$$

* Although in the figure x_3, the value of the abscissa, is a negative number according to Art. 2, yet x_3^2 being the square of a negative quantity is positive, and is therefore also the square of the length of the side OQ_3 of the triangle.

Similar equations can be derived for any point of the circle; and we see at once that the equation of the circle itself can be written simply:

$$(4) \qquad x^2 + y^2 = r^2,$$

an equation in x and y, which is satisfied when (and only when) the coördinates of any point whatever of the circle are substituted for x and y. This equation is satisfied by the pairs of coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3) . . . in the same way as the equations in Art. 3 were satisfied by the pairs of numbers there given.

Why does there exist a *single* equation which is satisfied by the coördinates of *all* the points of the circle? The reason is to be found in the fact that the relation between all the points of the circle and its center is governed by one and the same law. Every point in the circumference is equally distant from the center; what is true for the point P_1 and its coördinates x_1 and y_1 is also true of the coördinates of P_2 and P_3; it is even a matter of complete indifference which points we designate as P_1, P_2, P_3. In other words, all that we have to do in order to derive the equation which will be satisfied by the coördinates of all the points of the circle is to obtain the equation for any one of its points *arbitrarily* chosen; and this one point is *any* point, hence *every* point. Care must be exercised that the point chosen has no special or unusually simple position.

This principle is of great importance; from now on we shall make continual use of it in obtaining the equation of a curve.

We present the equation of the circle in another form also, *viz.* when its center does not lie at the origin of the coördi-

nates (Fig. 9). If the coördinates of the center, ON and MN, be equal to a and b respectively, let the coördinates of the point P, arbitrarily chosen, be

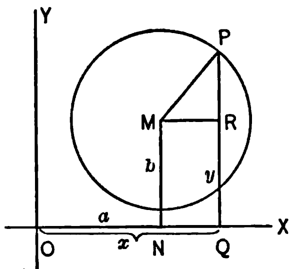

Fig. 9.

$$OQ = x$$

and $$PQ = y;$$

if MR is equal and parallel to NQ, we obtain directly from the right triangle MPR

$$\overline{MR}^2 + \overline{PR}^2 = \overline{MP}^2.$$

But since $$MR = NQ = OQ - ON = x - a,$$

and $$PR = PQ - RQ = PQ - MN = y - b,$$

we have, by substitution,

(5) $$(x - a)^2 + (y - b)^2 = r^2,$$

the *equation of the circle*.

Exercise. Discuss similarly various positions of P in the figure above, and also various positions of M in the different quadrants, as well as various magnitudes of the radius, including cases in which the circle cuts one or both axes; making a figure for each case, and, with due regard to algebraic signs, always reaching equation (5) as final result.

As a corollary to the above, we can deduce the formula which gives the *distance between two points* whose coördinates are known; equation (5) yields this formula at once. It states that the distance r of the point M from the point P is represented by the square root of the left-hand member of equation (5). If, for reasons of symmetry, we substitute a_1 and b_1 for x and y, our equation gives as the distance between two points with the coördinates (a, b) and (a_1, b_1)

(6) $$r^2 = (a_1 - a)^2 + (b_1 - b)^2.$$

EXAMPLE. The square of the distance between the points P_1 and P_2 (Fig. 10) whose coördinates * are (3, 4) and (2, 1) amounts to

$$(3-2)^2+(4-1)^2, \text{ or } 10.$$

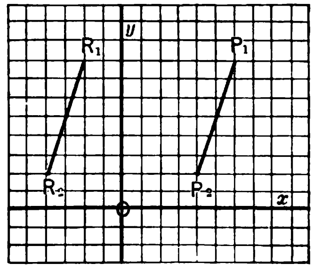

FIG. 10.

If we conceive the line P_1P_2 to be moved four units of length towards the left, so that it assumes the position R_1R_2, the coördinates of the points R_1R_2 are $(-1, 4)$ and $(-2, 1)$, and the square of the distance between them proves to be, as required,

$$(-1-(-2))^2+(4-1)^2$$
$$=(2-1)^2+(4-1)^2=10.$$

Our formula holds, therefore, even when the coördinates of the points have negative values. The reason for this lies, of course, in the fact that the same rules of calculation apply to both positive and negative quantities.

EXERCISES III

1. Find the equations of the circles having the following centers and radii (the point given and the number following it being respectively center and radius):

(i.) (3, 4), 6; (v.) (0, 5), 4; (ix.) $(-2, 0)$, 6;

(ii.) $(-2, 5)$, 2; (vi.) $(0, -2)$, $\frac{1}{2}$; (x.) (b, c), 2;

(iii.) $(-2, 4)$, $3\frac{1}{2}$; (vii.) (4, 0), $2\frac{1}{2}$; (xi.) (m, n), k;

(iv.) $(3, -1)$, 2; (viii.) (0, 0), 1; (xii.) $(4a_1, -2b)$, $3h$.

2. Find the distances between the following pairs of points:

(i.) (2, 3), (4, 5); (iv.) (8, 2), $(-4, 3)$;

(ii.) $(-2, 4)$, (1, 0); (v.) $(a, -a)$, $(b, 2b)$;

(iii.) $(1, -4)$, $(-2, -3)$; (vi.) $(a. b)$, (b, a);

(vii.) $(r, \sin\alpha)$, $(r, \cos\alpha)$.

---

* The unit of length is equal to two of the spaces into which the x-axis is divided in the figure.

3. Show that the coördinates of the middle point of the straight line joining the points (a, b) and (c, d) are

$$\frac{a+c}{2}$$

and

$$\frac{b+d}{2}.$$

(Construct figures variously, with given points lying in various quadrants.)

4. Find coördinates of the middle points of the lines joining each pair of points in **2**.

5. Find the equations of circles each passing through the point $(7, -3)$, and having as centers respectively the various points given in **2**.

ART. **5.** **The equation of the parabola.** Geometrically, *the* **parabola** *is the locus of all points which are equidistant from a fixed point and a fixed straight line.* If P (Fig. 11) be any point of the parabola, F the fixed point, d the fixed straight line, and PD the distance of the point P from d, the condition that defines the parabola is expressed by the equation

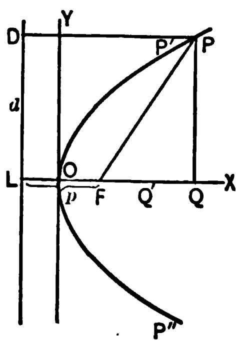

FIG. 11.

$$(1) \qquad PF = PD.$$

In order to express the equation of the parabola in the simplest possible form, we choose, as the x-axis, the perpendicular FL let fall from F on d, and the middle of the line FL, as the origin of the system of coördinates. The distance FL is denoted by p, and is called the **parameter** of the parabola. If x and y are the coördinates of the point P, it follows that

$$PD = OQ + \frac{p}{2} = x + \frac{p}{2};$$

(2) $$\overline{PF}^2 = \overline{FQ}^2 + \overline{PQ}^2 = \left(x - \frac{p}{2}\right)^2 + y^2;$$

and, by comparison with (1),

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \left(x + \frac{p}{2}\right)^2,$$

or $$x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4},$$

and, finally,

(3) $$y^2 = 2\,px.$$

This is the *equation of the parabola.*

We add that the point F is termed the **focus**, and the straight line d the **directrix** of the parabola.

Exercise. As in the preceding article, discuss various positions of P, making figure, and always reaching equation (3) as final result.

The question arises, how can the form of the parabola be deduced from its equation? It is perfectly evident that, if x has a negative value, y^2 must also be negative. But there are no real numbers whose square is negative, and hence the points of the parabola can lie only on the right side of the y-axis. If

$$x = 0, \text{ then } y = 0;$$

i.e. the parabola passes through the origin. If x be given any positive value, as, for instance,

$$x = OQ',$$

the equation furnishes two different values for y, but differing only in sign; the corresponding points are P' and P'', which are situated at the same distance from Q' above and

below the axis of abscissæ. This is true for every value of x; the points of the locus are therefore arranged in pairs symmetrically with reference to the axis of abscissæ; accordingly, the x-axis is denominated an **axis of symmetry** of the parabola.

We remark further that, if greater and greater values be given to x, the values of y increase continually; the farther the parabola extends from the x-axis, the more it spreads out. This gives us a preliminary idea of the form of the parabola.

On inspection, we see that the curve constructed in Art. 3 (p. 14) is a parabola, whose parameter is equal to unity.

ART. 6. **The equation of the straight line through the origin.** We first deduce the equation of the straight line on the assumption that it passes through the origin of a system of rectangular coördinates; in which case it may be defined as *a locus such that the angle formed with the x-axis by the (unterminated) straight line joining each point of the locus to the origin remains constant.* This angle, however, must be defined more exactly, since two straight lines can form different angles with each other. Accordingly, we establish the following general conditions:

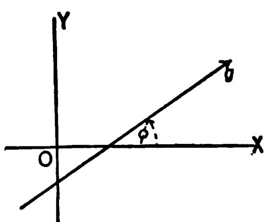

FIG. 12.

If the positive portion of the axis of abscissæ be rotated counter-clockwise about O, (Fig. 12) as a center and through an angle of 90°, it comes into the position of the positive portion of the axis of ordinates. This direction of rotation is called **positive**, and we are to understand by the angle which a straight line g makes with the x-axis, that angle through which the positive x-axis must be rotated

in the positive direction around the point of intersection of g and the x-axis until it coincides with g; in Fig. 12 this is the angle ϕ. The line g is capable of two directions, each measured from the vertex ϕ in opposite directions. The positive direction is measured from the vertex toward that part of the line g with which the positive part of the x-axis first coincides in its rotation counter-clockwise. (In the figure this is shown by an arrow.)

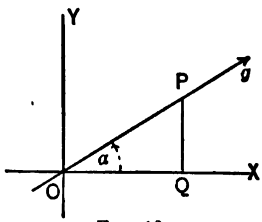

FIG. 13.

If g (Fig. 13) be the straight line whose equation we are to find, we have, to start with, the definition that the line connecting any point P with O always makes the angle α with the axis of abscissæ. If we designate by x and y the coördinates of P, it follows directly from the right triangle POQ that

$$\tan \alpha = \frac{PQ}{OQ} = \frac{y}{x};$$

or

$$y = x \tan \alpha, \tag{1}$$

which is the required equation of the straight line.

EXAMPLES

1. The equation $\qquad y - x = 0,$
or $\qquad y = x,$
represents a line for which $\qquad \tan \alpha = 1;$
that is, $\qquad \alpha = 45°.$
Likewise $\qquad y + x = 0,$
or $\qquad y = -x$
is a straight line for which $\qquad \tan \alpha = -1;$
that is to say, $\qquad \alpha = 135°.$

These two lines bisect the angles which the axes of the coördinates form with each other.

2. The equation $$y = 0$$
represents a straight line passing through O and having
$$\tan \alpha = 0,$$
and hence $$\alpha = 0;$$
in other words, the line is the axis of abscissæ. The law expressed by the equation
$$y = 0$$
means simply that the line is the locus of all points whose ordinates are equal to zero, and this is equivalent to saying that these points lie in the x-axis. In the same way
$$x = 0$$
is the equation of the y-axis; that is to say, the equation of the locus of all such points as have abscissæ equal to zero (p. 10).

3. The equation
$$y = \sqrt{3} \cdot x$$
represents a straight line passing through O, and making an angle of 60° with the x-axis.

ART. 7. **The equation of any straight line.** If the straight line (Fig. 14) has any position whatever with reference to the coördinate axes, and if, (i.) G is its point of intersection with the y-axis, (ii.) α is the angle which it makes with the x-axis, and (iii.) the distance OG is equal to b, we can define it by saying that the line connecting any of its points P with G has the same angle of inclination, α, with the x-axis. If PQ and GN be perpendicular and parallel respectively to the x-axis, the angle PGN is equal to α, and in the right-angled triangle PGN,

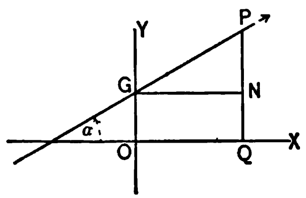

FIG. 14.

$$(1) \qquad \tan \alpha = \frac{PN}{GN} = \frac{y - b}{x},$$

where x and y are the coördinates of P; hence its follows that

$$(2) \qquad y = x \tan \alpha + b,$$

which is the required equation of the straight line. It is customary to denote tan α by m, so that the equation assumes the form,

$$(3) \qquad y = mx + b.$$

The equation of every straight line * is of this form, various equations differing from one another only in the values of m and b, which depend upon the positions of the lines. For example, if the straight line passes through the origin, its equation becomes simplified into

$$(4) \qquad y = mx,$$

quite in accordance with the foregoing results. Conversely, a straight line is represented by every equation of the form

$$y = mx + b,$$

no matter what the values of m and b may be. For there must always be a point G determined by b, whatever may be the number that b stands for; and likewise for every number m there must exist an angle α, so that

$$\tan \alpha = m;$$

if, then, we form the equation of the straight line, cutting off a distance b on the y-axis, and making an angle whose tangent is m with the x-axis, we shall have

$$y = mx + b.$$

---

* Provided the line is not perpendicular to the x-axis. We have already seen (Ex. 5, iii., p. 11) that the equations of all straight lines perpendicular to the x-axis are of the form $x = c$.

But this is precisely the equation under consideration, and we see, therefore, that it represents a straight line irrespective of the values of m and b.

We do not think it superfluous to furnish direct proof that our conclusions and results are not changed when b or m have negative values; that is, when the position of the straight line with reference to the coördinate axes leads to a figure apparently different. For the straight line drawn in Fig. 15, it follows from the triangle PGN that

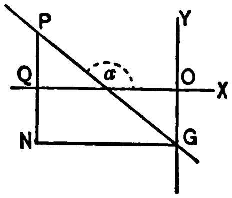

FIG. 15.

$$\tan 180 - \alpha = \frac{PN}{NG} = \frac{PQ + QN}{NG}.$$

But in this case x and b are negative quantities and accordingly, (p. 9), the length of QN is expressed by $-b$, and that of NG by $-x$.* Furthermore,

$$\tan 180 - \alpha = -\tan \alpha,$$

and hence
$$-\tan \alpha = \frac{y - b}{-x},$$

or
$$y = x \tan \alpha + b = mx + b.$$

The reason for the general validity of the results lies in the circumstance that the rules of calculation with positive and with negative quantities, as well as the trigonometric formulæ for acute and for obtuse angles, are the same; so that, even though the *figures* may be different for different positions of the straight lines, *their properties, their laws* — and everything hinges upon these alone — remain the same in all cases. In what follows we may accordingly omit calling special attention in each case to the generality of our formulæ.

Exercise. Deduce the equation

$$y = mx + b$$

from other possible figures, noting also that the figure is altered if, while the line remains unmoved, the variable point P be taken in a different quadrant.

---

* Regarded as coördinates, QN and NG represent negative numbers, *viz.* b and x. Regarded as purely geometric magnitudes, their lengths are positive, *viz.* $-b$ and $-x$. It often happens, as above, that we first discuss problems geometrically, all the lines involved being regarded as positive magnitudes, irrespective of position, and then express these positive magnitudes in terms of the coördinates (positive or negative) which the lines represent.

ART. 8. **Every equation of the first degree represented by a straight line.** We deduce from the foregoing the important conclusion that *every equation of the form*

(1) $$Ax + By + C = 0,$$

wherein A, B, and C are any positive or negative numbers whatever, is the *equation of a straight line.* For, dividing the equation by B* and transposing, we get,

$$y = -\frac{A}{B}x - \frac{C}{B};$$

and this is the equation for a straight line in which

(2) $$\tan \alpha = -\frac{A}{B},$$

and (3) $$b = -\frac{C}{B}.$$

We now see that the equation of p. 12,

$$x + y = 4, \text{ or } y = -x + 4,$$

represents a straight line, which cuts off on the axis of ordinates a distance equal to 4 units, and forms with the axis of abscissæ an angle such that $\tan \alpha = -1$, *i.e.* an angle of 135°, just as is shown in Fig. 6.

Equation (1), being general in character and containing only the first powers of x and y, is called **the general equation of the first degree**; from (3, p. 25) and (1) together we see that *the straight line is the graphic equivalent of the general equation of the first degree.* The equation of the first degree is accordingly often called the **linear** equation.

---

* This implies that B is not zero. When B is zero, the equation takes the form $x = -\frac{C}{A}$, or $x = c$, which has been discussed previously (pp. 11, 25).

EXAMPLES

1. In the equation

$$4x - 2y + 5 = 0,$$

$$b = \tfrac{5}{2},$$

and

$$\tan \alpha = 2;$$

(α being accordingly equal to about 63° 26′).

2. In the equation

$$x - 3y + 6 = 0,$$

$$b = 2,$$

and

$$\tan \alpha = \tfrac{1}{3},$$

(or $\alpha =$ about 18° 26′).

The two lines can be drawn by the aid of these data.

ART. 9. **The intercepts.** Use is often made of another method to find out the position of a straight line from its equation. The angle α is not employed in it, since angles are inconvenient in actual constructions; but any two points on the line are sought; and these determine the position of the straight line. The points which can be found most conveniently are those at which the lines intersect the axes. If the equation of the straight line is given in the general form

$$(1) \qquad Ax + By + C = 0,$$

these points are obtained in the following manner: The point of intersection with the axis of ordinates is the point whose abscissa is equal to zero; we find it by making x, in the above equation, equal to zero. This gives the point

$$(2) \qquad x = 0, \quad y = -\frac{C}{B}.$$

Similarly, the point of intersection with the axis of abscissæ is the point whose ordinate

$$y = 0;$$

this condition yields the equations

$$y=0, \quad x=-\frac{C}{A}. \tag{3}$$

The distances from the origin to the points of intersection with the axes are called the **intercepts** on the axes.

EXAMPLE. The points of intersection of the straight line

$$5x-7y+2=0,$$

with the axes are $(0, \frac{2}{7})$ and $(-\frac{2}{5}, 0)$, and the intercepts are $-\frac{2}{5}$ and $\frac{2}{7}$.

EXERCISES IV

1. Find the intercepts on the axes, and the tangent of the angle made with the x-axis by the straight lines which have the following equations:

(i.) $3x-2y+7=0$; (iv.) $3x=9y-2$; (vii.) $2y+3x=0$;
(ii.) $y=2x-5$; (v.) $2x=3y$; (viii.) $y+3x-5=0$;
(iii.) $2y+5x-4=0$; (vi.) $5x+9y+4=0$; (ix.) $4y-x+2=0$.

2. Examining Fig. 14, we see that the straight line starts from the first quadrant, passes *through* the second into the third; in Fig. 15 the line passes from the second, *through* the third, into the fourth quadrant. *Through* what quadrant does each line of exercise **1** above pass? (Answer by inspection of the equations.)

3. What are the equations of the straight lines parallel respectively to those of **1**, and

i. Passing through the origin;
ii. Having the intercept 5 on the y-axis?

ART. **10. Gay Lussac's Law.** According to the law of Gay Lussac, gases possess the following property: If their volume be kept constant, the pressure necessary to confine them to that constant volume increases proportionally to the temperature, and, indeed, if p_0 be the pressure at 0° Centigrade, the increase for one degree is $\frac{p_0}{273}$, and hence at $t°$ the pressure must be

$$p=p_0+\left(\frac{p_0}{273}\right)t=p_0\left(1+\frac{t}{273}\right).$$

If, to simplify matters, we assume the value of the pressure p_0 to be 1, this formula becomes,

$$p = 1 + \frac{t}{273}.$$

This equation is an equation of the first degree in t and p; if we substitute for p and t, y and x, respectively, so that the above equation becomes,

$$y = 1 + \frac{x}{273},$$

it may be represented by a straight line; this straight line (Fig. 16) is the graphic representation of the law, and shows with great clearness that the pressure varies continuously with the temperature.*

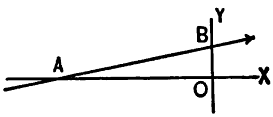

FIG. 16.

ART. **11. Problems on the straight line.** I. *What is the position of a straight line whose equation is*

$$(1) \qquad \frac{x}{a} + \frac{y}{b} = 1?$$

We determine as before the points at which it intersects the axes. In order to obtain the intercept on the y-axis, we put

$$x = 0$$

in the above equation, and find

$$y = b;$$

* In order to obtain the correct position of the straight line, OA has to be taken 273 times as great as OB. This is not, however, convenient for the sketch, and the above figure is but an approximate representation of the line. A similar method must always be employed when the numerical values of the coördinates are in too unfavorable relations for an accurate drawing.

let this point be B (Fig. 17). Similarly, A, the point of intersection with the x-axis is found to have the coördinates,

$$x = a,$$
$$y = 0.$$

The quantities a and b are, accordingly, the intercepts on the axes.

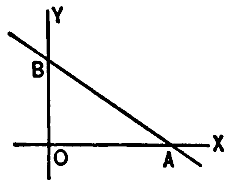

FIG. 17.

The form $$\frac{x}{a} + \frac{y}{b} = 1$$

is known as the **symmetric** equation of the straight line.

EXAMPLES

1. By writing the equation

$$x + y = 4$$

(p. 12) in the form

$$\frac{x}{4} + \frac{y}{4} = 1,$$

it appears that the straight line which it represents cuts off a distance 4 on each axis.

2. In the same way we find for the equation

$$4x + 3y - 2 = 0,$$

when transformed into

$$\tfrac{4}{2}x + \tfrac{3}{2}y = 1,$$

the intercepts $$a = \tfrac{2}{4} = \tfrac{1}{2},$$

and $$b = \tfrac{2}{3}.$$

II. *To determine the equation of a straight line, having a given direction with reference to the axis of abscissæ, and passing through a given point P, whose coördinates are x_1 and y_1.*

According to p. 25, the equation of every straight line has the form

(2) $$y = mx + b;$$

to find the equation of a *given* straight line, it is necessary to find the values of m and b which correspond to that line. Let, then, equation (2) be the equation of our straight line; then m is known, *viz.*

$$m = \tan \alpha,$$

α being the given angle, while b has a definite but as yet unknown value. Since the coördinates of the point P must satisfy the equation of the straight line, it follows that

$$(3) \qquad y_1 = mx_1 + b;$$

b alone is unknown in this equation, and can therefore be found from it and substituted in the first equation. By so doing we have, in principle, solved our problem. But we can obtain the required equation in a somewhat different way. To calculate b from equation (3), and to substitute its value in equation (2), is to *eliminate* b from equation (2) by means of equation (3). The simplest way to effect this elimination is to subtract one equation from the other; we then find

$$(4) \qquad y - y_1 = m(x - x_1),$$

and this is the required equation of the straight line.

III. *To find the equation of a straight line which passes through two given points whose coördinates are $x_1 y_1$ and $x_2 y_2$.*

The required equation of the straight line must have the form

$$(5) \qquad y = mx + b,$$

where m and b are definite, although as yet unknown, quantities whose values are to be calculated. Since the coördinates x_1, y_1 and x_2, y_2 satisfy the equation of the straight line, we obtain

(6) $$y_1 = mx_1 + b;$$

(7) $$y_2 = mx_2 + b,$$

and these are the two equations from which we are to calculate the values of m and b, and then substitute them in equation (5). This means, in other words, that we must eliminate m and b from our three equations, (5), (6), and (7). The simplest way is the following: We subtract the third equation from the first and from the second equation, and thus obtain

$$y - y_2 = m(x - x_2),$$

$$y_1 - y_2 = m(x_1 - x_2);$$

and by dividing the first of these equations by the second, we get finally

(8) $$\frac{y - y_2}{y_1 - y_2} = \frac{x - x_2}{x_1 - x_2}$$

as the required equation.*

EXAMPLES

1. The equation of the straight line that passes through the points (2, 1) and (– 3, 4) reads:

$$\frac{y-4}{1-4} = \frac{x+3}{2+3},$$

or, in a simplified form, $3x + 5y - 11 = 0$.

2. The equation of the straight line passing through the points (3, 2) and (– 3, – 2) is

$$2x - 3y = 0;$$

the line accordingly passes through the origin.

---

* Of course the equation can also be put into a form from which the values of m and b can be directly read off. A simple transformation of equation (8) gives:

$$y = \frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

IV. *To find the equation of a straight line, given the length, p, of the perpendicular on it from the origin, and the angle α which that perpendicular makes with the x-axis.*

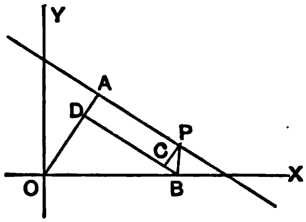

FIG. 18.

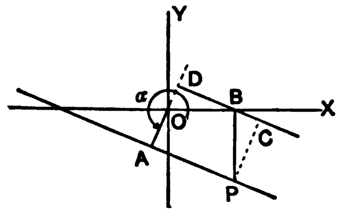

FIG. 19.

In both figures (18, 19), $OA = p$; $BOA = \alpha$.*

Then $\quad OA = OD + DA$

$\qquad\qquad = OD + PC.$

But $\quad PBC = \alpha,$

and hence,

$PC = PB \sin \alpha = y \sin \alpha.$

Likewise,

$OD = OB \cos \alpha = x \cos \alpha.$

$$OA = DA - OD$$
$$= PC - OD.$$

But

$$PBC = DOB = \alpha - 180°,$$

and hence,

$$PC = PB \sin PBC$$
$$= PB \sin(\alpha - 180°)$$
$$= -\,y \sin(\alpha - 180°)$$
$$= y \sin \alpha.$$

Likewise,

$$OD = OB \cos DOB$$
$$= x \cos(\alpha - 180°)$$
$$= -\,x \cos \alpha.$$

(9) $\qquad\qquad$ $\therefore\ \boldsymbol{p = y \sin \alpha + x \cos \alpha.}$

This is known as the **normal** equation of the straight line.

---

* As OA is a *terminated* portion of a straight line, we regard as the angle between OA and the x-axis, the angle through which the positive portion of the x-axis revolves in the positive sense until it comes for the first time into coincidence with OA.

ART. 12. Concerning the nature of a general equation.

In the various forms of the equation of the straight line, which we have considered, there enter constant coefficients. When these constants are given different values, different straight lines are represented, and every possible straight line may be represented by giving the constant coefficients proper values. We may start from a given straight line fulfilling given conditions (such as to pass through two given points), and set ourselves the problem to find the equation which will represent this line, *i.e.* to find the particular values which must be given to the constants of the general equation (say $y = mx + b$) in order that it may represent the line under consideration. We have solved several problems of this kind. In doing so we regarded the constants as unknown quantities to be determined. This notion of constant quantities which may vary (and which may indeed assume all possible values) is sometimes perplexing to the student. The fact is that the coefficients are constant for every specific line, but vary from line to line, while the variables x and y assume for each line all the values compatible with the relation established between them by the equation. The notion of arbitrary constants in a general equation is of such fundamental importance that we add an illustration which may make it clearer.

A printed form of mortgage can be bought at the law stationer's. It contains blanks for the names of the mortgagor and mortgagee, for the exact description of the piece of land mortgaged, for the consideration and the amount of the mortgage, for the date when it is executed, and the time it has to run. Until these blanks are properly filled out it is no particular mortgage, but it may become *any* mortgage by filling out the blanks suitably. Yet all the properties of a mortgage *as* a mortgage can be learned from this printed form; its general legal aspects and force can be determined as well, perhaps better, than if it were specialized into a particular mortgage. The blank form is *the general mortgage* — all the characteristics common to all mortgages can be learned from it, — and when this *general mortgage* is once thoroughly understood it is a trifling matter to draw up a specific mortgage. We may also discuss, if we please, the various kinds of mortgages that may arise from various styles of filling out the blanks, such as farm mortgages, mortgages on vacant city lots, on improved city real estate and the like.

The case of a general equation, for instance $y = mx + b$, is quite analogous. This contains two blanks, one denoted by m, for the tangent of α, the angle which the straight line represented by the equation makes with the x-axis; the other denoted by b, for the length of the intercept on the

y-axis. When the blanks m and b are not filled out, *i.e.* have no specific numerical value, this equation represents no particular straight line, but it may represent *any* straight line by filling out the blanks suitably. It is the *general* equation of the straight line; from it in the blank form

$$y = mx + b$$

all the properties common to all straight lines can be learned, and when this general form is thoroughly understood, the equation of specific straight lines can be written out at will. We may also discuss the classes of straight lines we obtain by filling out blanks in various styles; thus, if we fill the blank b with zero, the line will pass through the origin, no matter how the blank m is filled out. The equation

$$y = mx$$

is then the blank form for the equations of all straight lines passing through the origin, or it is the general equation of all such lines. Similarly

$$y = x + b$$

is the general equation of all lines making an angle of 45° with the axis of x.

We have spoken above of $\quad y = mx + b$

as the general equation of the straight line. This is not meant to imply that there may not be other general equations of the straight line. Just as there may be more forms than one for a mortgage, such that all possible mortgages may be drawn up according to either one, so there may be, and in fact are, more forms than one, of equations such that all possible straight lines may be represented by any one of them, by filling up the blank coefficients properly.

EXERCISES V

1. Plot the straight lines represented by the following equations:

(i.) $3x + 2y = 7$;
(ii.) $y = 2x - 2$;
(iii.) $\frac{y}{4} - \frac{x}{2} = 1$;
(iv.) $2x - 3y = 1$;
(v.) $\frac{x}{4} + 5y = 1$;
(vi.) $x \cos 40° + y \sin 40° = 3$;
(vii.) $x \cos 112° + y \sin 112° = 2$;
(viii.) $x \cos 243° + y \sin 243° = 5$.

2. Find the general equation of all straight lines through the point (2, 4), likewise find the general equation of all straight lines through the point $(-1, 5)$, and also of those through the point $(-2, -2)$.

3. Find the equation of the particular straight line through each of the points of **2** above, which makes an angle of 120° with the x-axis; likewise of the straight line through each point which makes an angle with the x-axis whose tangent is -2; likewise of those which make an angle with the y-axis whose tangent is $\frac{1}{4}$.

4. What is the general equation of all straight lines:

(i.) Parallel to the x-axis?

(ii.) Parallel to the y-axis?

(iii.) Through the origin?

(iv.) Making an angle of 45° with the x-axis?

(v.) Making an angle of $-45°$ with the x-axis?

(vi.) Parallel to

(a) $y = 4x - 2$, (c) $x = y$,

(b) $2x + 3y = 6$, (d) $y = -4x$?

(vii.) At the distance 3 from the origin?

5. Write the equation of the straight lines passing through the point (2, 1) and parallel respectively to the first five lines in **1** above.

6. Find the equations of the straight lines passing through the following pairs of points:

(i.) (1, 2), (0, 1); (iv.) (−6, 4), (3, 5);

(ii.) (−4, 3), (−2, −1); (v.) (a, a), (b, b);

(iii.) (1, −1), (−1, 1); (vi.) (a, b), (b, a);

(vii.) (a, b), (c, d).

7. Three of the vertices of a parallelogram are (−4, 1), (−1, −6), (2, 3). Find the equations of its four sides.

8. Deduce the *normal* equation of the straight line from a figure in which the straight line passes

(i.) through the second quadrant;

(ii.) through the fourth quadrant;

(iii.) through the origin.

9. Denoting by d the distance of a straight line from the origin, and by A the angle which the perpendicular from the origin on the straight line makes with the x-axis, write the equations of the lines having:

$d = 5$	2	4	3	2
$A = 120°$	45°	225°	270°	330°

10. Starting from the general equation in the *normal* form, find the equations

(i.) of the bisectors of the quadrants;

(ii.) of all lines parallel to the x-axis;

(iii.) of all lines parallel to the y-axis.

11. A triangle has as its vertices the points $(2, 3)$, $(1, 7)$, $(-4, 2)$. Find the equations of its sides.

12. A right triangle has the vertices of its acute angles in the points $(4, 6)$, $(-2, -5)$, and the other sides parallel to the axes. Find the equations of its sides, and its area.

13. Find the equation of the circle which has the points $(3, 5)$, $(-4, -3)$ as extremities of a diameter. Use this result to name the third vertex of four right triangles having the points $(3, 5)$ and $(-4, -3)$ as extremities of their hypotenuse, and the third vertex lying in turn in each of the four quadrants. How many solutions are possible? Make a second choice of the four third vertices, so that they shall be the four corners of a rectangle. Find the equations of the diagonals of your rectangle.

ART. **13.** **Two straight lines.** If two straight lines are given, we are, above all, interested in knowing their point of intersection and the angle which they make with each other.

Let the equations of the two straight lines be

$$(1) \qquad y = mx + b \text{ and } y = m'x + b'.$$

Each equation is satisfied by a boundless number of pairs of values of x and y, and indeed, each one is satisfied by the coördinates of any of its points. These pairs of values are generally different, but there is necessarily one and only one pair among them that satisfies both equations, and that is the one which corresponds to the point of intersection of the lines. In order to find this pair of values, we have to determine by the ordinary methods the values of the unknown quantities x and y which satisfy both equations simultane-

ously. The solving of two equations of the first degree with two unknown quantities means, then, geometrically speaking, the finding of the point of intersection of the two lines, which are represented by the two equations.

Example. The three straight lines

$$x + 7y + 11 = 0,$$
$$x - 3y + 1 = 0,$$
$$3x + y - 7 = 0$$

define a triangle whose vertices are the points of intersection of these lines taken in pairs, viz.: $(2, 1)$, $(3, -2)$, $(-4, -1)$.

The angle δ which two straight lines form with each other is to be understood as being the angle which their positive directions form. If α and α' denote the angles which the two lines make with the axis of abscissæ, the value of δ is

$$\delta = \alpha' - \alpha.$$

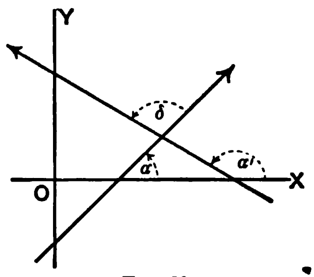

Fig. 20.

Accordingly,

$$\tan \delta = \tan(\alpha' - \alpha)$$

(2) $$= \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha' \tan \alpha}.$$

If we now substitute m for $\tan \alpha$, and m' for $\tan \alpha'$, we have

(3) $$\tan \delta = \frac{m' - m}{1 + mm'}.$$

Should the straight lines be parallel, then δ as well as $\tan \delta$ must be equal to zero; that is to say,

(4) $$m = m'.$$

If the two straight lines are perpendicular to each other, the magnitude of δ is 90°, while $\tan\delta$ becomes infinitely large; the denominator of the above fraction must accordingly be equal to zero,

$$(5) \qquad 1 + mm' = 0;$$

or

$$m' = -\frac{1}{m},$$

and only when this condition is fulfilled can the two lines be perpendicular to each other.

Example. The angle δ of the two straight lines

$$3x + y - 7 = 0$$

and

$$x - 3y + 1 = 0$$

is found to be 90° (the triangle mentioned above is therefore right-angled at the point (2, 1) of intersection of these two lines).

Remark. We have just seen that two equations of the first degree in x and y determine the position of the point at the intersection of the straight lines in question. Since when a and b are the coördinates of a point P, this can be expressed by the equations

$$(6) \qquad x = a, \quad y = b,$$

the question immediately arises as to whether these two equations can be interpreted in the way indicated above. As a matter of fact this is the case.

The equation $y = b$ is according to p. 25, the equation of a straight line for which $m = 0$, and hence $\tan\alpha$ (as well as α itself) is equal to zero; that is, the line is parallel to the axis of abscissæ; further (Fig. 14, p. 24), it passes through the point G of the axis of ordinates, for which $OG = b$; it is accordingly parallel to the x-axis and is at a distance b from it. The equation $y = b$ is the locus of all points whose ordinates are equal to b, and all of them lie upon this parallel to the x-axis. In a similar manner, it follows that $x = a$ represents a straight line which runs parallel to the y-axis at a distance a from it; and these two lines intersect in that point whose coördinates are defined by the equations (6).

Geometrically, then, the use of coördinates is tantamount to regarding every point of a plane as the point of intersection of two straight lines which are parallel to two fixed axes. If all possible parallels to the x-axis and all those to the y-axis be thought of, one of each of these sets of parallels passes through any given point, and the distances of these lines from the axes are the coördinates of the points in which they intersect.

EXERCISES VI

1. Find the intersections of the following pairs of lines:

(i.) $2x - 3y + 5 = 0,\ y - 4x + 7 = 0;$

(ii.) $\frac{x}{2} + \frac{y}{3} = 1,\ 4y = x;$

(iii.) $4x - 9y = 2,\ 3x + 2y = -5;$

(iv.) $ax + by = 1,\ bx + ay = 1.$

2. (i.) What is the general equation of all straight lines through the point $(-1, 7)$?

(ii.) Through the point $(a, -3a)$?

(iii.) What is the general equation of all straight lines parallel to $y = 4x - 3$?

3. Show that the general equations of all straight lines, perpendicular to

$$y = mx + b$$

is

$$y = -\frac{x}{m} + c.$$

4. (i.) Write the general equation of all straight lines perpendicular to each of the lines of **1** above.

(ii.) Write the equation of the perpendicular from the origin to each of the lines of **1**.

(iii.) Find the tangent of the angle between each of the pairs of lines in **1**.

5. Find the vertices of all triangles formed by the four lines:

$$x = -y,$$
$$y = \frac{x}{2} + 3,$$
$$y + 10x + 18 = 0,$$
$$x = -16 - 4y.$$

6. In each triangle of **5** find:

(i.) The equation of the perpendicular from one vertex (any one) on the opposite side.

(ii.) The coördinates of the foot of the perpendicular.

(iii.) The length of the perpendicular.

(iv.) The length of the side to which the perpendicular is drawn.

(v.) The area of the triangle.

7. In any one of the triangles of **5** verify by forming the equations of the perpendicular bisectors of the sides, the geometrical theorem that the perpendicular bisectors of the sides of a triangle meet in a point.

8. By considering the general triangle, whose sides are:

$$ax + by + c = 0,$$
$$dx + ey + f = 0,$$
$$gx + hy + k = 0.$$

prove generally the theorem verified in **7.**

9. Which of the triangles that can be formed by combining any three of the following lines is right angled?

$$x = 2y + 7,$$
$$2y + x + 5 = 0,$$
$$3y + 6x = 15,$$
$$y = 2x + 1.$$

10. What are the equations representing the sides of the general right triangle?

11. Verify in one of the right triangles found in **9** that the line joining the middle of the hypotenuse to the opposite vertex divides the triangle into two isosceles triangles.

12. By considering the general right triangle as found in **10** prove generally the theorem verified in **11.**

ART. **14. The equation of the ellipse.** We define the ellipse as *the locus of a point which moves so that the sum of its distances from two fixed points has a constant value.* This constant value we indicate by $2a$.

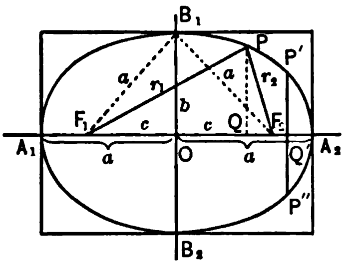

FIG. 21.

We take (Fig. 21) the line connecting the two fixed points F_1 and F_2 as the axis of abscissæ, and the perpendicular erected at the middle of F_1F_2 as the axis of ordinates. The distance F_1F_2 is designated by $2c$, and r_1 and r_2 represent the distances of any point in the ellipse, as P from F_1 and F_2, so that

(1) $$r_1 + r_2 = 2a;$$

furthermore, it is apparent that

(2) $$2c < 2a.$$

If x and y are the coördinates of the point P, it is seen from the figure that

(3) $$r_1^2 = (c + x)^2 + y^2;$$

(4) $$r_2^2 = (c - x)^2 + y^2.$$

If these values of r_1 and r_2 be substituted in equation (1), the equation showing the relation between x and y is obtained; that is, the equation of the ellipse. This is best done as follows:

In order to avoid calculations with radical signs, we raise equation (1) to the second power, obtaining:

$$r_1^2 + 2r_1r_2 + r_2^2 = 4a^2,$$

$$r_1^2 + r_2^2 - 4a^2 = -2r_1r_2.$$

By squaring the latter equation, we find

(5) $$(r_1^2 + r_2^2)^2 - 8a^2(r_1^2 + r_2^2) + 16a^4 = 4r_1^2r_2^2,$$

or, transposing and rearranging,

(6) $$(r_1^2 - r_2^2)^2 - 8a^2(r_1^2 + r_2^2) + 16a^4 = 0.$$

From equations (3) and (4),

$$r_1^2 + r_2^2 = 2(x^2 + y^2 + c^2),$$

$$r_1^2 - r_2^2 = 4cx;$$

and by substitution in (6),

$$16c^2x^2 - 16a^2(x^2 + y^2 + c^2) + 16a^4 = 0,$$

or (7) $$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2).$$

Dividing both sides by $a^2(a^2-c^2)$, we have

$$(8) \qquad \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

If B_1 is a point of the axis of ordinates such that

$$B_1F_1 = B_1F_2 = a;$$

and if we designate OB_1 by b, we obtain from the right triangle B_1F_1O,

$$(9) \qquad a^2-c^2=b^2;$$

and by substitution our equation passes into the form

$$(10) \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is *the equation of the ellipse.* The ratio $\frac{c}{a}$ is known as the **eccentricity** of the ellipse.

ART. **15.** **The form of the ellipse.** We endeavor next to get an idea of the form of the ellipse. Giving x any definite value, as, for example, $x = OQ$, the corresponding values of y are found from equation (10) to be

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}.$$

For every value of x there are two values of y that differ only in sign, and hence determine two points of the locus, P' and P'', lying symmetrically with reference to the axis of abscissæ. *The x-axis is accordingly an axis of symmetry for the ellipse;* since the equation of the ellipse has the same form for both y and x, it follows that *the y-axis is also an axis of symmetry for the ellipse.* The axes divide the ellipse into four congruent quadrants; the discussion of one of these quadrants will suffice.

We ask, therefore, how will y change when x increases from zero? If $x = 0$, then $y = \pm b$; that is, the point B_1, as well as B_2, which is symmetrical with it, are points of the ellipse. If x increases, $1 - \frac{x^2}{a^2}$ (and with it y) becomes smaller and smaller, until, when $x = a$, both $1 - \frac{x^2}{a^2}$ and y become equal to zero; the values $x = a$, $y = 0$ determine a point of intersection of the ellipse with the x-axis; the symmetrically located point A_1 is likewise a point of the ellipse. If x still continues to increase, $\frac{x^2}{a^2} > 1$, and $1 - \frac{x^2}{a^2}$ becomes negative; and there is, therefore, no real value of y corresponding to a value of $x > a$. Consequently, the points of the ellipse all lie within the strip bounded by two lines drawn parallel to the axis of ordinates and through the points A_1 and A_2. In a similar manner it follows that they also all lie within a strip, which two lines form drawn through B_1 and B_2 and parallel to the axis of abscissæ. Hence the ellipse lies within a rectangle with sides $2a$ and $2b$, as shown in the figure.

A_1A_2 is termed the **major axis**, and B_1B_2, the **minor axis** of the ellipse. The lengths of these axes are $2a$ and $2b$; a and b themselves are the **semiaxes**. The points A_1, A_2, B_1, B_2 are named **vertices**, and the points F_1 and F_2, **foci**.

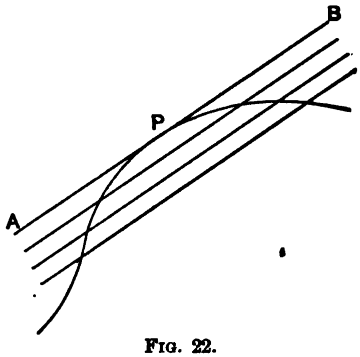

FIG. 22.

ART. 16. **Problems concerning the ellipse.** Geometrically defined, the tangent to a curve is that straight line which is the limiting position which a secant line approaches as

two consecutive points of intersection approach coincidence. Thus (Fig. 22), as the secant line moves parallel to itself, the two points of intersection move closer and closer together, and finally coincide at P. The line APB may therefore be regarded as a special case of a secant line in which two intersections coincide. The same result may also be obtained (Fig. 23) by keeping one of the points of intersection fixed, and revolving the line about it as a pivot until a second point of intersection comes to coincidence with it. It follows from what we have said that to find *the equation of a tangent to a curve* whose equation we know, we must find the equation of a secant line in which two points of intersection have been made to coincide. We take up a few problems which will sufficiently explain the method.

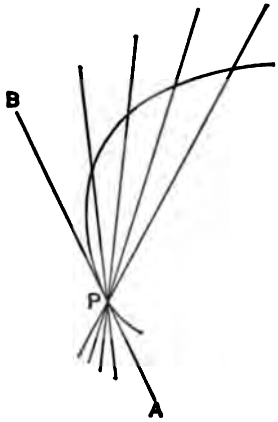

FIG. 23.

I. *To find the equation of the secant line through the points* (x_1y_1) *and* (x_2y_2) *on the ellipse,*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We have seen that, regarding these points as *any* points, without reference to the ellipse, the equation of the straight line through them is (p. 33),

$$(1) \qquad \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

But if the points lie on the ellipse, their coördinates must satisfy the relations

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\,;\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1\,;$$

subtracting and transposing,

$$\frac{y_2^2-y_1^2}{b^2}=-\frac{x_2^2-x_1^2}{a^2}. \tag{2}$$

Multiplying equations (1) and (2), member by member, we have

$$\frac{(y-y_1)(y_2+y_1)}{b^2}=-\frac{(x-x_1)(x_2+x_1)}{a^2}, \tag{3}$$

or (4)
$$\frac{x_2+x_1}{a^2}x+\frac{y_2+y_1}{b^2}y=\frac{x_1(x_2+x_1)}{a^2}+\frac{y_1(y_2+y_1)}{b^2}$$
$$=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}$$
$$=1+\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}.$$

This is the equation of the straight line through two points of the ellipse. If these two points are brought to coincidence, the secant line becomes the tangent. Letting (x_2, y_2) coincide with (x_1, y_1), we have

$$\frac{2\,x_1}{a^2}x+\frac{2\,y_1}{b^2}y=1+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2,$$

or (5)
$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1.$$

This is, therefore, the equation of the tangent to the ellipse at the point x_1y_1 on it.

A simpler, and at the same time more general method of determining the tangent will be established in the Calculus.

II. ***To find the condition that a given straight line may touch the ellipse.***

Let the equation of the straight line be

$$y = mx + c.$$

Regarded geometrically, the straight line will, in general, intersect the ellipse in two distinct points, and in the particular cases in which these points are coincident, the given line will be tangent to the ellipse. We shall first find the intersections, and then determine under what conditions they are coincident. The coördinates of the points of intersection must satisfy both the equations

$$y = mx + c \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence for these points we have

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

or (6) $$(b^2 + a^2m^2)x^2 + 2\,a^2cmx + a^2c^2 - a^2b^2 = 0.$$

The roots of this equation will be the values of the abscissæ of the points of intersection, and for each abscissa the equation $y = mx + c$ will determine one ordinate. There are, then, two points of intersection, and if the roots of the equation in x are equal, these points will have the same coördinates; *i.e.* will coincide. But the condition that the equation (6) shall have equal roots is *

(7) $$(2\,a^2cm)^2 - 4\,(b^2 + a^2m^2)\,(a^2c^2 - a^2b^2) = 0,$$

* Formula 73, Appendix.

which reduces to $\qquad a^2m^2 + b^2 - c^2 = 0,$* $\qquad$ or

$$c = \pm\sqrt{a^2m^2 + b^2}. \tag{8}$$

Whenever c and m are so related that the relation just written is satisfied, then, and only then, the line

$$y = mx + c$$

is tangent to the ellipse. In other words, the line

$$y = mx \pm \sqrt{a^2m^2 + b^2} \tag{9}$$

is tangent to the ellipse, no matter what value m may have, and will represent all possible tangents by giving m all possible values. It is the *general* equation of the tangent to the ellipse.

EXAMPLES

1. *To find the tangent from the point* (3, 2) *to the ellipse*

$$\frac{x^2}{5} + \frac{y^2}{11} = 1.$$

Here $a^2 = 5$, $b^2 = 11$, and hence, by equation (9) above,

$$y = mx \pm \sqrt{5\,m^2 + 11}$$

is tangent to the ellipse for all values of m.

We have yet to choose m so that the line passes through the point (3, 2). Since this is to be the case, we must have

$$2 = 3\,m \pm \sqrt{5\,m^2 + 11}.$$

Solving this equation, we find

$$m = \tfrac{7}{2} \text{ or } -\tfrac{1}{2}.$$

Substituting this value of m above, we have

$$y = \frac{7\,x}{2} \pm \frac{17}{2}.$$

---

* To reach this form, we divide by a^2, thus making the assumption that a is not zero. If a were zero, the ellipse would consist of a single point.

Of these, only the line $$y = \frac{7x}{2} - \frac{17}{2}$$

passes through the point (3, 2).* Similarly, the value $m = -\frac{1}{2}$ gives

$$y = -\frac{x}{2} + \frac{7}{2}$$

as the second tangent from the point (3, 2) to the ellipse.

2. *What is the equation of the tangent to the ellipse*

$$\frac{x^2}{4} + \frac{y^2}{3} = 1,$$

at the point $(1, -\frac{3}{2})$? Here

$$x_1 = 1,\ y_1 = -\tfrac{3}{2},\ a^2 = 4,\ b^2 = 3,$$

and we have as the equation of the tangent,

$$\frac{x}{4} - \frac{y}{2} = 1.$$

3. *Find the equation of the tangents to the ellipse*

$$\frac{x^2}{2} + \frac{y^2}{7} = 1,$$

which make an angle of 45° *with the* x-*axis.*

In this case $m = \tan 45° = 1$, $a^2 = 2$, $b^2 = 7$, and the lines are $y = x + \sqrt{2 + 7}$, or $y = x + 3$ and $y = x - 3$ are the two tangents which make an angle of 45° with the x-axis.

ART. 17. The auxiliary circle; the directrix; the eccentricity.

I. Let us consider an ellipse (Fig. 24) with the semi-axes a and b, and a circle having the major axis of the ellipse as

---

* The general equation of the tangent to the ellipse shows that there are always *two* tangents having the same direction (as determined by m), which is plain geometrically. In our problem, we determine the directions of the particular tangents which pass through the given point. Of course, only *one* of the two tangents, having the direction found, will pass through the point.

diameter; and let P and P' be points of the ellipse and of the circle lying on the same perpendicular to the x-axis. These points have the same abscissa x, but different ordinates, which we designate by y and y'. The equation of the ellipse is then satisfied by x and y, and that of the circle by x and y'; *i.e.*

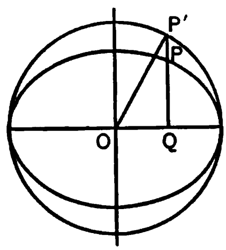

FIG. 24.

$$x^2 + y'^2 = a^2,$$

or (1) $$\frac{x^2}{a^2} + \frac{y'^2}{a^2} = 1,$$

and (2) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Whence by subtracting and transposing,

$$\frac{y'^2}{a^2} = \frac{y^2}{b^2},$$

(3) $$\frac{y'}{y} = \frac{a}{b},$$

or, $$\frac{P'Q}{PQ} = \frac{a}{b};$$

accordingly *the corresponding ordinates of the ellipse and the circle are in a constant ratio.* This circle is called the **auxiliary circle** of the ellipse.

II. *To determine the distances of any point P of the ellipse from the foci F_1 and F_2.*

According to p. 43, the distance $\overline{PF_2}^2$ or r_2^2 is:

(4) $$r_2^2 = (c - x)^2 + y^2.$$

The value of y^2 as deduced from the equation of the ellipse is

$$y^2 = b^2 - \frac{b^2}{a^2}x^2.$$

But (p. 44) $$b^2 + c^2 = a^2;$$

further, $$x^2 - \left(\frac{b^2}{a^2}\right)x^2 = \left(\frac{c^2}{a^2}\right)x^2;$$

hence we get $$r_2^2 = a^2 - 2\,cx + \frac{c^2}{a^2}x^2,$$

or (5) $$r_2^2 = \left(a - \frac{c}{a}x\right)^2.$$

In the same way it is found that

(6) $$r_1^2 = (c + x)^2 + y^2 = \left(a + \frac{c}{a}x\right)^2.$$

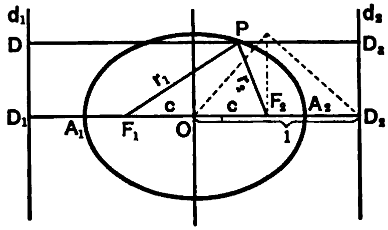

FIG. 25.

Therefore the values of the distances PF_1 and PF_2 are

(7) $$r_1 = a + \frac{c}{a}x,$$

and $$r_2 = a - \frac{c}{a}x.$$

From this an important result may be obtained.

We put $$\frac{a^2}{c} = l,$$

that is, $$c : a = a : l,$$

l is a distance defined by a and c which may be constructed as the hypotenuse of a right triangle in which a is one side and c is its projection upon the hypotenuse, just as is shown

by the figure. We lay off OD_2 equal to l, and draw through D_2 a line d_2 parallel to the axis of ordinates; if a perpendicular from P be dropped on this line, it is easily seen that

(8) $$PD_3 = l - x = \frac{a^2}{c} - x.$$

Writing r_2 in the form

(9) $$r_2 = \frac{c}{a}\left(\frac{a^2}{c} - x\right);$$

and substituting from (8) we have

$$r_2 = \left(\frac{c}{a}\right) PD_3,$$

or (10) $$\frac{PF_2}{PD_3} = \frac{c}{a}.$$

The straight line d_2 is called the **directrix** of the ellipse; the foregoing equation states in regard to it that the *ratio of the distances of any point of an ellipse from the focus and from the directrix has the fixed value* $\frac{c}{a}$.

As a consequence of the symmetry of the ellipse there must be another directrix d_1 belonging to the focus F_1, the distance of which from O is also equal to l, and for which the same law holds true.

The ratio $\frac{c}{a}$ has already been defined (p. 44) as the eccentricity of the ellipse. Since $c = \sqrt{a^2 - b^2}$, and hence $c < a$, the eccentricity of the ellipse is always less than unity. As

$$OD_2 = l = \frac{a^2}{c},$$

we can readily write the equation of the directrix, *viz.* $x = \frac{a^2}{c}$ $\left(\text{or } x = -\frac{a^2}{c}, \text{ for the other directrix}\right)$.

EXERCISES VII

Note. Throughout this set of exercises, the axes of the ellipse are taken to be the coördinate axes.

1. Construct the equations of the ellipses having as semi-major and semi-minor axes respectively the following pairs of values:

(i.) 3, 2; (ii.) 4, 1; (iii.) 6, 2; (iv.) 5, 5.

2. Find the coördinates of the foci of the ellipses in **1**.

3. Find the equations of ellipses which have respectively the following semi-major axes and foci:

(i.) 3, (2, 0)
(ii.) 5, (4, 0);
(iii.) 6, $(-3, 0)$;
(iv.) 7, $(-1, 0)$;
(v.) 3, (0, 0);
(vi.) 2, $(-\frac{1}{2}, 0)$;
(vii.) h, $(-k, 0)$.

4. Find the equations of ellipses having respectively the first of each of the following pairs of points as focus, and passing through the second:

(i.) (2, 0), (2, 3);
(ii.) $(-3, 0)$, $(3, -\frac{16}{5})$;
(iii.) (5, 0), $(-2, -5)$;
(iv.) $(f, 0)$, (c, d).

5. Find the equations of ellipses, referred to their axes as axes of coördinates, and passing respectively through the following pairs of points (cf. method of p. 33, Eqs. 5 . . 8):

(i.) (3, 1), (1, 5);
(ii.) $(4, -2)$, $(-3, -3)$;
(iii.) $(-2, 6)$, $(4, -2)$;
(iv.) (1, 3), $(6, -1)$.

6. Find the equations of the tangent from the point $(-1, 6)$ to the ellipse

$$\frac{x^2}{36} + \frac{5y^2}{81} = 1.$$

7. Find the equation of the tangent to the ellipse

$$2x^2 + 8y^2 = 1,$$

at the point $(-\frac{1}{2}, \frac{1}{4})$.

8. To which of the following ellipses, if any, is the line $x = 11 - 8y$ tangent?

(i.) $3x^2 + y^2 = 4;$
(ii.) $2x^2 + 5y^2 = 1;$
(iii.) $8x^2 + 16y^2 = 176;$
(iv.) $\dfrac{x^2}{25} + \dfrac{y^2}{100} = 1.$

9. Find the general equation, representing all tangents to each of the ellipses of **8**.

10. Find the equations of each tangent to every ellipse of **8** which makes equal intercepts on the axes.

11. Determine the eccentricity and the equation of the directrix of each of the following ellipses:

(i.) $$\frac{x^2}{5} + 3y^2 = 1;$$

(ii.) $$\frac{x^2}{6} + \frac{y^2}{4} = 2;$$

(iii.) $$5x^2 + 10y^2 = 1.$$

12. Find the relation between the corresponding abscissæ of points on the ellipse and on the circle whose diameter is the minor axis of the ellipse.

ART. **18. The equation of the hyperbola.** We define the hyperbola as *the locus of a point which moves so that the difference of its distances from two fixed points is constant.* Let the value of this constant difference be $2a$.

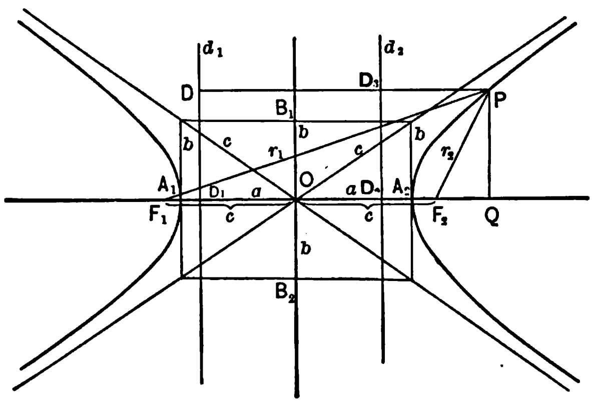

FIG. 26.

We take (Fig. 26) the line connecting the fixed points F_1 and F_2 as the axis of abscissæ, and the perpendicular at its middle point O as the axis of ordinates. We designate

by $2c$ the distance F_1F_2, and by r_1 and r_2, respectively, the distances from F_1 and F_2 of the point P (lying on the hyperbola), so that, by definition,

(1) $$r_1 - r_2 = 2a.$$

Since in every triangle the difference of two sides is less than the third side, it follows that

(2) $$2a < 2c.$$

If the coördinates of P be x and y, it is seen from the figure that

(3) $$r_1^2 = (x + c)^2 + y^2;$$

(4) $$r_2^2 = (x - c)^2 + y^2.$$

By squaring (1) and transposing we get

$$r_1^2 + r_2^2 - 4a^2 = 2r_1r_2,$$

and by squaring again,

(5) $$(r_1^2 + r_2^2)^2 - 8a^2(r_1^2 + r_2^2) + 16a^4 = 4r_1^2r_2^2.$$

Equations (3), (4), (5) are identical with (3), (4), (5) on p. 43, and by proceeding just as we did there, we obtain

(6) $$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

In the case of the hyperbola, however, $a < c$, as seen above; therefore $a^2 - c^2$ is negative. Putting

(7) $$a^2 - c^2 = -b^2,$$

we obtain the *equation of the hyperbola in its final form:*

(8) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

ART. **19. The form of the hyperbola.** As on p. 44, here also both the axes of coördinates are axes of symmetry. To show the course of the hyperbola in the first quadrant, the equation may be written

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1;$$

it is apparent that the right member is negative so long as $x < a$; hence for all such values of x the corresponding values of y are imaginary, *i.e.* no corresponding geometric point of the hyperbola exists. No part of the hyperbola lies in the strip between the perpendiculars to the x-axis at $x = a$ and $x = -a$. If

$$x = a,$$

the right member is zero, and the pair of values

$$x = a,$$

and $$y = 0$$

determines the point A_2 of the positive x-axis through which the hyperbola passes; the point A_1 of the negative x-axis $(OA_1 = -a)$ also belongs to the hyperbola. If x increases still more, *i.e.* if x has any value greater than a, y is always real and increases continuously with x; the hyperbola extends to an unlimited distance in each quadrant as is seen in the figure.

The line A_1A_2 is termed the **real axis** of the hyperbola and the axis of y — from analogy — the **imaginary axis.** The points A_1 and A_2 are called **vertices**; a is the *real and b the imaginary semi-axis;* F_1 and F_2 are again named **foci.**

ART. **20. The directrix of the hyperbola.** *To calculate for the hyperbola the distances of any of its points from the foci.*

According to p. 56 we have for the distance PF_1 or r_1

(1) $$r_1^2 = (x + c)^2 + y^2.$$

As in the analogous case for the ellipse (II., p. 51), we obtain

(2) $$r_1^2 = \left(\frac{c}{a}x + a\right)^2,$$

whence

(3) $$r_1 = \frac{c}{a}x + a,$$

and similarly,

(4) $$r_2 = \frac{c}{a}x - a.$$

The value of r_2 may also be written thus:

(5) $$r_2 = \frac{c}{a}\left(x - \frac{a^2}{c}\right).$$

We put

(6) $$\frac{a^2}{c} = l,$$

i.e. $$c : a = a : l,$$

l being defined by a and c, just as the analogous distance on p. 52; here, however, $a > l$, since $c > a$. We lay off OD_2 (Fig. 26) equal to l, draw through D_2 the line d_2 parallel to the y-axis, and let fall upon it from P the perpendicular PD_3; we then have, as before,

$$PD_3 = x - l = x - \frac{a^2}{c},$$

and by substituting this in (5) we find

(7) $$\frac{PF_2}{PD_3} = \frac{c}{a}.$$

As in the case of the ellipse the straight line d_2 is called the **directrix** of the hyperbola; *the distance of any point of the hyperbola from the focus and from the directrix are in the*

constant ratio $c:a$. A directrix d_1 belongs to the focus F_1, for which the same law holds. The ratio $c:a$ is called the **eccentricity** of the hyperbola.

ART. **21. The equilateral hyperbola and its asymptotes.** If it be assumed that the axes $2a$ and $2b$ of an ellipse are equal to each other, the ellipse passes into the circle; the circle is therefore the simplest case of the ellipse. If $a=b$ in the equation of the hyperbola, a remarkably simple hyperbola is obtained, which is called the **equilateral hyperbola.** Its equation is

(1) $$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1,$$

or $$x^2-y^2=a^2.$$

We term the lines bisecting the angles between the coördinate axes the **asymptotes** of this hyperbola, and propose:

To find the equation of the equilateral hyperbola when its asymptotes are taken as axes.

As an aid to the solution of the problem we present the following preliminary considerations. We take any two straight lines, passing through O (Fig. 27) and at right angles with each other, as the axes of a new system of coördinates. Let the coördinates of any point P referred to them be ξ and η. We draw PQ perpendicular to the axis of x and PR perpendicular to the axis of ξ, so that $RPQ = ROQ = \alpha$ and

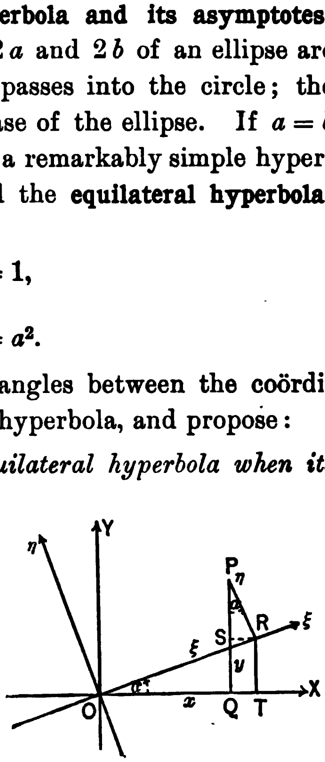

FIG. 27.

(2) $$OQ = x, \quad PQ = y,$$
$$OR = \xi, \quad PR = \eta,$$

and further draw RT perpendicular and RS parallel to the axis of x. It then follows that

$$x = OQ = OT - TQ = OT - RS,$$

$$y = PQ = PS + QS = PS + RT.$$

But from the triangles ORT and PRS,

$$OT = \xi \cos \alpha,$$

$$RS = \eta \sin \alpha,$$

$$RT = \xi \sin \alpha,$$

$$PS = \eta \cos \alpha,$$

and hence,

(3) $$x = \xi \cos \alpha - \eta \sin \alpha,$$

$$y = \xi \sin \alpha + \eta \cos \alpha;$$

and these are the equations which show how the coördinates of a point P referred to one system of axes are related to its coördinates in the other system.

Applying this to the asymptotes taken as new axes (Fig. 28), we see that $\alpha = -45°$ (p. 22), and therefore

$$\cos \alpha = \sqrt{\tfrac{1}{2}},$$

and $$\sin \alpha = -\sqrt{\tfrac{1}{2}};$$

we accordingly obtain for this special case the equations

(4) $$x = \xi\sqrt{\tfrac{1}{2}} + \eta\sqrt{\tfrac{1}{2}},$$

$$y = -\xi\sqrt{\tfrac{1}{2}} + \eta\sqrt{\tfrac{1}{2}},$$

whence,

(5) $$x - y = 2\xi\sqrt{\tfrac{1}{2}},$$

$$x + y = 2\eta\sqrt{\tfrac{1}{2}}.$$

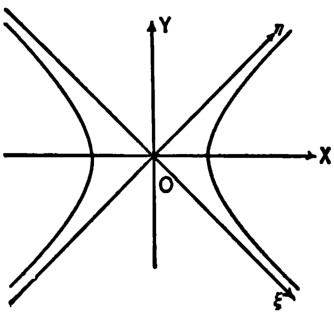

FIG. 28.

If P is a point of an equilateral hyperbola, its coördinates must satisfy the equation

$$x^2 - y^2 = a^2,$$

which we can also write in the form

$$(x + y)(x - y) = a^2. \tag{6}$$

By introducing the previous values, we get as the equation of the same point P, referred to the axes ξ, η,

$$2\,\xi\eta = a^2, \tag{7}$$

and since the coördinates of any point of the hyperbola satisfy this relation, *this equation is the equation of the equilateral hyperbola referred to its asymptotes as coördinate axes.*

We now see that Boyle's Law is represented graphically by a hyperbola; for if we substitute p for ξ and v for η, and put

$$\frac{a^2}{2} = 1,$$

the equation becomes

$$pv = 1.$$

The following geometric property of the asymptotes is interesting. By writing equation (7) in the form

$$y = \frac{a^2}{2\,x},$$

we see that the smaller y is the greater is x; *i.e.* the hyperbola approaches nearer and nearer the axis of x the farther it extends, but never reaches it, no matter how large x may become. (An abbreviated form of this statement often used is that the hyperbola reaches the axis of x only if $x = \infty$. This mode of abbreviation will be discussed in the next chapter.) A similar statement is true of the axis of y. For

this reason, the axes of x and y are called ***asymptotes*** * of the hyperbola; the hyperbola approaches nearer and nearer to both lines the farther they extend, but never reaches them.

In all the preceding articles the axes have been supposed to be at right angles to each other. It is possible to treat all the problems which we have hitherto discussed without making this assumption, but as the results when the axes are not at right angles with each other are of much less importance, we pass them by with this mention. *In the exercises which follow, the axes are always supposed to be rectangular.*

EXERCISES VIII

1. Show (in a manner analogous to that explained in the case of the ellipse) that the equation of the tangent to the hyperbola, at the point x_1y_1 on it, is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

2. Show likewise that

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

is tangent to the hyperbola for all values of m.

3. Using the results of the previous exercises, find the equations of:

(i.) The tangent to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

at the point $(4, -6)$.

(ii.) The tangents to the same hyperbola from the point $(-1, 1)$.

(iii.) The tangents to the same hyperbola from the origin.

(iv.) The tangents from the point $(6, 2)$. Interpret this result geometrically.

4. Find the tangents from the origin to the hyperbola

$$x^2 - y^2 = 4.$$

5. Find the equation of the equilateral hyperbola

$$x^2 - y^2 = 9$$

referred to its asymptotes as axes.

---

* Grk. ἀσύμπτωτος, not falling together.

ART. **22. Transformation of coördinates.** We solved the problem treated in the last article, by introducing a new system of coördinates. The introduction of new systems of coördinates is often of great importance. The position of the axes is indeed arbitrary, but it is readily seen that there will usually be for every curve, or geometric construction, some preferable position. Generally it cannot be determined which position this is, until the equation of the curve when referred to an arbitrarily assumed system of axes has been deduced. We must accordingly establish formulæ that will enable us to pass from equations referred to one system of coördinates to equations referred to another system. To begin with, we assume the axes of both systems to be parallel to one another. For example, in Fig. 29, let

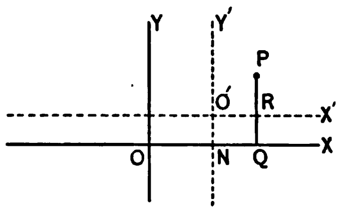

FIG. 29.

(1) $$X = O'R, \quad Y = PR,$$

be the coördinates of the point P referred to the axes $O'X'$ and $O'Y'$. Furthermore, let

(2) $$x = OQ, \quad y = PQ,$$

be the coördinates of the point P in the system of coördinates whose origin is at O; finally let

(3) $$a = ON, \quad b = O'N,$$

be the coördinates of the point O' in the latter system. Then, between the coördinates x, y and X, Y of the point P, we have the equations

(4) $$X = x - a, \quad Y = y - b;$$

these equations are true of *every* point.

The equation of the circle for the coördinates X, Y is

$$(5) \qquad X^2 + Y^2 = r^2,$$

r being the radius; this equation is true for every point, P, of the circle. If we substitute for X and Y their values as given in (4), we get

$$(6) \qquad (x-a)^2 + (y-b)^2 = r^2;$$

this equation is satisfied by the coördinates x, y, of any point, P, of the circle, and is therefore the equation of the circle with the new system of axes. We obtained the same equation in a different way on p. 18.

If we take new axes having the same origin as the original ones, but different directions, the formulæ (3), (p. 60), are to be employed, *viz.*:

$$(7) \qquad x = \xi \cos\alpha - \eta \sin\alpha; \quad y = \xi \sin\alpha + \eta \cos\alpha;$$

by multiplying them by $\cos\alpha$ and $\sin\alpha$ respectively, and adding, and also performing the same operations with $-\sin\alpha$, $+\cos\alpha$,* we find

$$(8) \qquad \xi = x\cos\alpha + y\sin\alpha; \quad \eta = -x\sin\alpha + y\cos\alpha.$$

The coördinates x, y and ξ, η can therefore be expressed, each in terms of the others, in exactly the same way. We can pass from one system of coördinates to any other, having a new origin and different directions of axes, by carrying out two transformations of coördinates, one after the other.

The transformation of coördinates is of very great utility. By its aid it can be proved that (disregarding some exceptions) *every equation of the second degree represents an* **ellipse,**

* Formula 28, Appendix.

a **hyperbola,** *or a* **parabola.** This is done by transforming the system of coördinates so that the most general equation of the second degree, *viz.*,

$$ax^2 + 2\,bxy + cy^2 + 2\,dx + 2\,ey + f = 0, \tag{9}$$

passes into one of the forms which we have already found for the equations of the ellipse, the hyperbola, and the parabola.

EXERCISES IX

1. Find the equation of the circle:

$$x^2 + y^2 = r^2,$$

(i.) If the new origin is at the upper end of the vertical diameter and (*a*) the new axes are parallel to the old; (*b*) if the new axes make an angle of 45° with the old.

(ii.) If the new origin is at the right end of the horizontal diameter, and (*a*) the new axes are parallel to the old, (*b*) the new axes make an angle of 60° with the old.

(iii.) If the axes are the tangent at the lower extremity of the vertical diameter and the tangent at the left extremity of the horizontal diameter.

2. Show that in transforming from one system of rectangular coördinates to any other, the *degree* of the equation of any curve is not altered.

3. Find the equation of the ellipse referred to the major axis and a tangent at the left vertex.

4. Find the equation of the hyperbola referred to its real axis and the tangent at the right vertex.

5. Find the equation of $4\,x^2 - 9\,y^2 = 36$, referred to axes having the same origin as the old axes and making an angle of $-60°$ with them.

6. Find the equation of

$$x^2 - 5\,xy + y^2 + 8\,x - 20\,y + 15 = 0,$$

for new axes with origin at $(-4, 0)$ and

(i.) parallel to the original axes;

(ii.) making an angle of $-45°$ with the original axes.

7. Find the equation of

$$36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0,$$

referred to new axes with origin at $(2, -3)$ and

(i.) parallel to the original axes;

(ii.) making with original axes an angle whose tangent is $-\frac{4}{3}$.

8. Find the equation of

$$y^2 - 4y - 5x = 0,$$

referred to parallel axes with origin at $(-4, 2)$.

9. Find the equation of

$$16x^2 + 25y^2 + 32x - 100y - 284 = 0,$$

referred to parallel axes with origin at $(-1, 2)$.

ART. **23. Van der Waals's equation.** The equations of the curves thus far considered have been of only the first or second degree. Although it is beyond the scope of this book to discuss curves with equations of higher orders than the second, yet we mention at least one example of such curves.

The equation for Boyle's Law (p. 3) does not hold true when the pressure upon a gas exceeds certain limits. For the case of strongly compressed gases, van der Waals has proposed a celebrated equation, which commonly goes by his name; *viz.*:

$$(1) \qquad \left(p + \frac{a}{v^2}\right)(v - b) = 1;$$

in it a and b are positive constants, characteristic of the gas under consideration.

By taking the volume v and the pressure p as coördinates, we can represent graphically* the law formulated in equation (1).

* Since when we multiply out, a term (pv^3) appears, which is of the fourth degree in p and v together, the curve is said to be of the fourth order (or a quartic curve).

We now discuss this law briefly.

I. If we allow the mass of gas to occupy a large volume, that is, if we make v very large, the value of $\frac{a}{v^2}$ in equation will become very small, in practice, inappreciable; $v - b$, likewise, will not differ appreciably from v, so that we have, approximately,

$$pv = 1; \tag{2}$$

in other words, in the case of highly rarefied gases van der Waals's equation passes over into that for Boyle's Law.

II. If, however, v is not very large, that is, if the gas is not in a condition of considerable rarefaction, the influence of the constants a and b becomes appreciable.

If we make v very small by increasing the pressure p, equation (1) in which p, v, a, b are all positive, shows that v will approach b in value constantly, until, when p is enormously great, v approximately coincides with b. The constant b is the limit to the smallness of the volume which the gaseous mass can be made to assume through an increase of pressure; according to the above equation, a smaller volume than b is impossible.

The condition of affairs is made most evident by a graphic representation. For carbon dioxide (carbonic acid gas),

$$a = 0.00874; \ b = 0.0023;$$

hence

$$\left(p + \frac{0.00874}{v^2}\right)\left(v - 0.0023\right) = 1. \tag{3}$$

The pressure is reckoned in atmospheres; if we put $p = 1$, we obtain from the above equation a value for the volume, which is easily found to be 0.9936; (*i.e.* the unit of volume is a little larger than that which the mass of gas

under consideration occupies when subject to a pressure of one atmosphere).

The values of p and v, given in the following table, determine, when plotted, the curve shown in Fig. 30.

v	p	v	p
0.1	9.4	0.008	38.8
0.05	17.5	0.005	20.9
0.015	39.9	0.004	42.0
0.01	42.6	0.003	45.7

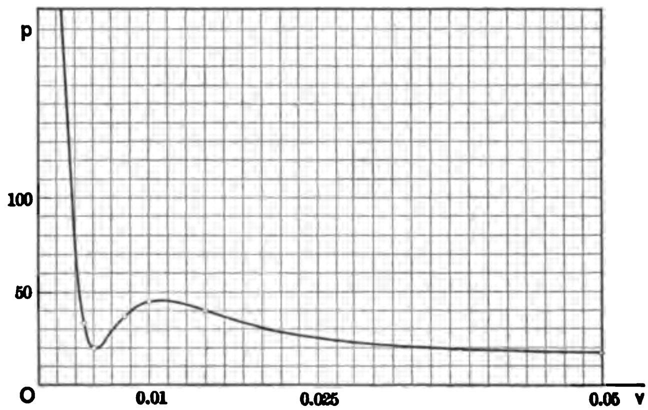

FIG. 30.

The equation

$$\left(p+\frac{0.00874}{v^2}\right)\left(v-0.0023\right)=1$$

is true only when the temperature is 0°; for carbon dioxide at the temperature t, van der Waals gives the equation

$$\left(p+\frac{0.00874}{v^2}\right)\left(v-0.0023\right)=1+\frac{t}{273};$$

if we assign to t in this equation different values (as 13.1, 21.5, etc.), we can draw, in a way similar to that above, a curve corresponding to each value of t. This group of curves (Fig. 31) gives us a clear view of the behavior of carbon dioxide under the most various conditions of pressure, volume, and temperature. By its consideration van der Waals was enabled to draw very far-reaching conclusions about the behavior of matter in a state of considerable condensation, both in the gaseous and liquid condition.

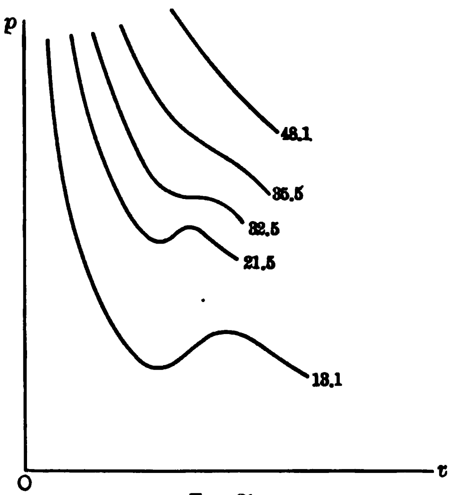

FIG. 31.

ART. **24. Polar coördinates.** The method of determining points in a plane by means of the coördinates which were defined (pp. 7–10), is not the only method possible. On the contrary, there may be a countless number of such methods, one other of which is of sufficient importance to require mention here.

If we conceive a number of circles to be drawn around O (Fig. 32), and if we draw through O any number of straight lines, a series of points of intersection is obtained, each of which points, as, for instance, P_1 or P_2, has its position determined, when we know its distance r_1 or r_2 from O, and the angle ϕ_1 or ϕ_2, which the line P_1O or P_2O makes with a fixed axis OX. As on p. 9, here also we find that

any point P corresponds to a pair of numbers; that is, to the length of r and the magnitude of ϕ, and, conversely, if a pair of numbers, as r and ϕ, is given, one point P can always be found whose position in the plane is defined by r and ϕ. The quantities r and ϕ are called the **polar coördinates** of the point P.

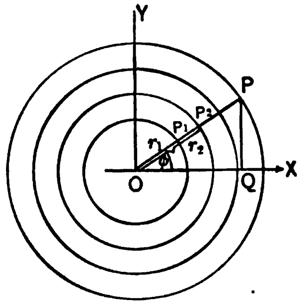

FIG. 32.

To determine the position of points in the plane by polar coördinates, we make use of two systems of lines; a system of concentric circles and a system of straight lines passing through their center. In the case of rectangular coördinates, we made use of two systems of straight lines respectively parallel. The use of two systems of lines is, in essence, the *general principle underlying every type of coördinates* employed in higher mathematics.

The relations between polar coördinates and rectangular coördinates, for which OX is the axis of abscissæ and O the origin, are seen from a consideration of triangle OPQ (Fig. 32) to be

(1) $$x = r\cos\phi,\ \ y = r\sin\phi;$$

(2) $$x^2 + y^2 = r^2.$$

By means of these equations it is possible to calculate the rectangular coördinates when the polar coördinates are known, and *vice versa*, the polar coördinates, when the rectangular coördinates are given.

ART. **25. The equations of the ellipse, the parabola, and the hyperbola in polar coördinates.** In Arts. 14, 5, and 20 it has been shown that *the ellipse, the parabola, and the hyperbola are each the locus of a point, which moves so that its*

distances from a fixed point (focus) and a fixed line (directrix) are in a constant ratio (eccentricity). For the ellipse this ratio is less than 1 (since $c < a$), for the hyperbola it is greater than 1 (since $c > a$), and for the parabola it is equal to 1 (since both distances are equal in the parabola). We deduced this property from certain definitions of these curves. We might, however, have set out with it as *definition*, in which case we should have deduced the previous definitions as properties of the curves. This would, indeed, have been a more general treatment, since the three curves would have been comprised under one definition, and it would have appeared from the outset that the three curves are all varieties of one general type. In the course of our study of the curves, we have seen this in connection with the eccentricity, and also in connection with the degree of the equations of the curves, all of the equations being of the second degree, and together constituting the totality of all curves whose equation is of the second degree. We mention further that these curves are all varieties of the plane sections of a circular cone, whose sides are produced in both directions without limit. The section is an ellipse, a parabola, or an hyperbola, according as the angle between the cutting plane and the axis of the cone is greater than, equal to, or less than, the angle between the axis and the edge of the cone. For this reason these curves are often called **conic sections**.

To deduce the equations of the conic sections in polar coördinates, we use as definition the property mentioned above. We designate (Fig. 33) the distance of the fixed point F_1 from the fixed straight line d_1 by p, and the eccentricity by e, and proceed to derive the equation of all three of the curves by one process. We take F_1 as the origin in

the polar coördinates, and the perpendicular F_1L, let fall from F_1 on d_1, as the axis; its positive half shall be that which does not intersect the right line d_1; we have then

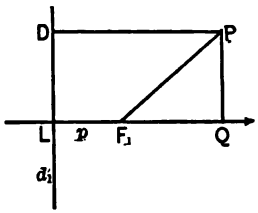

FIG. 33.

(1) $$\frac{PF_1}{PD} = e.$$

If r and ϕ be the polar coördinates of P, and PQ be a perpendicular from P on the axis, then

(2) $$PD = LF_1 + F_1Q = p + r\cos\phi,$$

and by substitution we obtain the required equation in the form

(3) $$\frac{r}{p + r\cos\phi} = e,$$

whence

(4) $$r = \frac{ep}{1 - e\cos\phi}.$$

For the ellipse (pp. 52–53)

$$e = \frac{c}{a}, \quad p = l - c = \frac{a^2}{c} - c = \frac{a^2 - c^2}{c} = \frac{b^2}{c},$$

so that its equation is

(5) $$r = \frac{\frac{b^2}{a}}{1 - \frac{c}{a}\cos\phi}.$$

For the hyperbola (pp. 58–59)

$$e = \frac{c}{a}, \quad p = c - l = c - \frac{a^2}{c} = \frac{c^2 - a^2}{c} = \frac{b^2}{c},$$

and hence its equation becomes likewise

$$(6) \qquad r = \frac{\frac{b^2}{a}}{1 - \frac{c}{a}\cos\phi}.$$

Since in the parabola $e = 1$, its equation is simply

$$(7) \qquad r = \frac{p}{1 - \cos\phi},$$

where p is the parameter of the parabola.

The fact that the equations of the ellipse, the parabola, and the hyperbola, when expressed in polar coördinates, have the same form, is of great importance in astronomy, particularly in the determination of the paths of comets. Every comet describes an ellipse, a parabola, or a hyperbola, of which the sun is a focus. Equation (4) is the equation of the comet's path, and the quantities p and e occurring in it are to be determined by observations on various positions of the comet in the heavens. From the value of e, it is known whether the comet describes an ellipse ($e < 1$), a parabola ($e = 1$), or an hyperbola ($e > 1$). In the first case the comet moves periodically around the sun, but in the other cases it is only a transient guest of our solar system.

Five observations are sufficient to determine the orbit. Two positions of the comet, together with that of the sun, determine the plane in which the comet moves, and the three other positions are needed to determine the three quantities remaining unknown, *viz.* the direction of the axis, p and e.*

* A sixth observation is needed, if we wish to determine the position of the comet in its orbit.

ART. **26. The Spiral of Archimedes.** We shall now show how curves, whose equations in rectangular coördinates are complicated, can be represented by polar coördinates in an extremely simple form. Such curves are, for instance, the spirals, of which that known as the spiral of Archimedes is an example. Its equation is

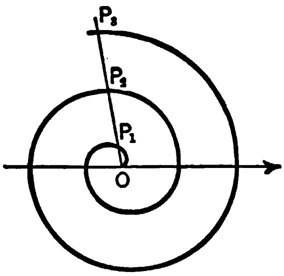

FIG. 34.

(1) $$r = a\phi.$$

In discussing this curve we shall find it most convenient not to measure the angle ϕ in degrees, but in circular measurement.* If P_1 and P_2 (Fig. 34) be two points of the spiral which belong to angles ϕ_1 and ϕ_2, differing by 2π, they lie upon one and the same straight line passing through O. The equations

(2) $$r_1 = a\phi_1, \quad \text{and} \quad r_2 = a\phi_2$$

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* In mathematical computations, angles are usually measured in degrees; that is, the unit is $\frac{1}{360}$ of the angular magnitude about a point. In theoretic mathematics, however, it has been found advantageous to introduce another unit of angular measurement; namely, that in which the unit is the **radian,** or the angle measured by an arc equal in length to the radius. This system of measurement, known as **circular measure,** is described in detail in works on trigonometry.

We have the relation $360° = 2\pi$ radians, which enables us to pass from degrees to radians, and *vice versa*. No symbol has been generally introduced for the unit of the circular measurement, but when degrees are not expressed, radians are understood as the unit. Thus the angles ϕ, $\frac{\pi}{2}$ are the angles which contain ϕ and $\frac{\pi}{2}$ radians respectively; ϕ and $\frac{\pi}{2}$ are merely numbers.

We recall also that in trigonometry we have extended our ideas about angles so as to treat of angles of any magnitude, positive as well as negative, and have defined the trigonometric functions for all such angles.

hold for the points in question, and, by subtraction, we find

(3) $$r_2 - r_1 = a(\phi_2 - \phi_1) = 2\,a\pi.$$

Carrying these considerations farther, we find without trouble that there is a countless number of points P_1, P_2, P_3, P_4, $\cdots$, lying on every straight line passing through O, which belong to the angles ϕ_1, $\phi_2 = \phi_1 + 2\,\pi$, $\phi_3 = \phi_1 + 4\,\pi$, $\phi_4 = \phi_1 + 6\,\pi$, $\cdots$; the distances of these points from O are given by the equations $r_2 = r_1 + 2\,a\pi$, $r_3 = r_1 + 4\,a\pi$, $r_4 = r_1 + 6\,a\pi$, $\cdots$. Since this is true of every straight line radiating from O, the spiral consists of a boundless number of revolutions which wind around the center, always keeping at a distance $2\,a\pi$ from one another. Inasmuch as for $\phi = 0$, $r = 0$, the spiral has its beginning in O.

ART. 27. **Concerning imaginary points and lines.** We began this subject by establishing a correspondence between pairs of numbers (called coördinates) and points. This correspondence was such that every pair of real numbers determines one, and only one, point of the plane, and *vice versa*. We have grown accustomed consequently instead of saying "the point whose coördinates are a and b," to say as abbreviation "the point (a, b)." We may regard the pair of numbers itself as a point — an *algebraic* "point," having the corresponding geometric point as its graphic representation. We have considered also various loci or graphs, made up of the totality of all points whose coördinates satisfy a certain equation given in each case. We determine whether or not a certain point lies on a given curve by determining whether or not its coördinates satisfy the equation of the curve. POINT LIES ON CURVE and COÖRDINATES SATISFY EQUATION are perfectly equivalent. We now extend our definitions.

We define *any* pair of numbers (regarded as coördinates) as a "point"; *i.e.* an algebraic point. If one or both of the numbers are imaginary, we call the point an **imaginary algebraic point.** This is purely an algebraic conception and has no geometric representation. If both numbers are real, we call the point a **real algebraic point,** and it has a geometric representation. We now say any point (real or imaginary) lies on a curve if the coördinates of the point satisfy the equation of

the curve. This convention enables us to state general algebraic theorems in geometric language. Thus, we may say:

Every straight line intersects every circle in two points.

This means neither more nor less than:

The equations $x^2 + y^2 = r^2$ *and* $ax + by + c = 0$ *have always two common solutions (real or imaginary).*

We have seen that the *equation of the first degree is the algebraic equivalent of the straight line.* This was proved with the tacit assumption that the coefficients of the equation are all real. If any of the coefficients become imaginary, the equation has no longer any geometric equivalent in our system of coördinates. However, the same type of relation exists between the two numbers of every pair which satisfies the equation (*viz.*, the ordinate is a multiple of the abscissa plus a constant), and therefore we now say that *every* equation of the first degree is the algebraic equivalent of a straight line. If the coefficients are all real, the straight line is *real*, and has a graphic representation. If any of the coefficients are imaginary, the straight line is *imaginary*, and has no graphic representation. Similarly, we speak of *imaginary* curves of every species.

CHAPTER II

CONCERNING LIMITS

ART. 1. **Constants, variables, and limits.** A **variable** quantity is one which may assume different values (usually a boundless number of them) in the same discussion. The coördinates x and y of a point on a curve are examples, and the law according to which they vary is expressed by the equation of the curve; in the case of the straight line by

$$y = mx + b.$$

A **constant** quantity is one which retains the same value throughout the same problem or discussion. The slope of a straight line and its intercept on the axis of y are examples of constants. They are denoted in the equation above by m and b.

A variable quantity which is considered to be quite arbitrary, and to which may be assigned any value at will, is called an **independent variable.** Illustrations of an independent variable are the radius of a circle, the area of a square, in short, any variable quantity which is quite arbitrary.

If we determine that an independent variable shall assume all possible values which differ less and less from a constant quantity l, and that this difference shall become small at will, we say that we "let the variable approach the **limit** l." If x denote the independent variable, this process of approaching the limit is indicated by $x \doteq l$. The sign $\doteq$ means to "approach the limit," and $x \doteq l$ is read, "x

approaches the limit l." As the variable is entirely independent, we can let it approach any limit we please, or which the conditions of any particular problem may lead us to select. No question can then ever arise as to what limit an independent variable approaches; for it may approach any limit we choose to set. When an independent variable is defined, there is nothing more to be said about it, and it accordingly offers little to interest us.

The case is changed, however, when we consider some second variable quantity whose value depends upon that of the independent variable, and is determined by it. Such a variable is called a **dependent variable**. The values of a dependent variable are not at all arbitrary, but are fixed as soon as the values of the independent variable are fixed. Given the radius of a circle as independent variable, the area of the circle is a dependent variable; likewise the circumference and the diameter are dependent variables; given the area of a square, the side and the diagonal are dependent variables. The question at once arises: If we let the independent variable approach some limit l, what does the dependent variable do? Clearly the dependent variable may change values in consequence of changes in the value of the independent variable. These changes of value may or may not be such that, as the values of the independent variable differ less and less from its limit, the corresponding values of the dependent variable differ less and less from some fixed quantity, in such a way that the latter difference may be made as small as we wish, by making the former difference small enough. If this is the case, the fixed quantity from which the values of the dependent variable differ less and less is called the **limit** of the dependent variable.

ART. 2. **Illustrations of limits.** I. Consider a triangle of altitude unity, and variable base. The area of the triangle is dependent upon the length of the base. The length of the base is the independent variable, the area is the dependent variable. If, now, the length of the base be made to differ less and less from 2, the area differs less and less from unity, and can be made to differ from unity just as little as we please by taking the base little enough different from 2. Thus the area differs from unity by less than $\frac{1}{100}$, whenever the length of the base is less than $\frac{2}{100}$ different from 2.

II. In the same triangle, let the variable base approach the limit zero. If we take the base small enough, the area can be made small at will. Hence the area approaches the limit zero as the base approaches zero.

III. Given a circle of radius unity, and a concentric circle of variable radius $x \leqq 1$. Now, considering the area of the circular ring between the two circles, we have the area equal to $\pi(1 - x^2)$. As x becomes more and more nearly equal to unity, the area becomes more and more nearly equal to zero. By taking x sufficiently little different from 1, the area may be made to differ as little as we please from zero. Hence, again, as x approaches the limit 1, the area approaches the limit zero.

IV. Considering $2x - 1$, and letting $x \doteq -4$, we see that $2x - 1$ may be made to differ little at will from -9 by taking x sufficiently little different from -4. Hence $2x-1$ approaches the limit -9 as x approaches the limit -4.

V. Consider the fraction $\dfrac{2}{x^2+1}$; as x approaches 3, the denominator approaches 10, and the fraction approaches $\frac{1}{5}$. By taking x sufficiently little different from 3, the fraction

may be made to differ little at will from $\frac{1}{5}$. Hence $\frac{1}{5}$ is the limit of $\dfrac{2}{x^2+1}$ as x approaches 3.

ART. 3. **Definition of limit.** As the precise definition of a limit when met for the first time is somewhat difficult of comprehension or application, we begin with several definitions, which, though loose, are often given:

1. *The* **limit** *of a variable is a constant quantity from which the variable may be made to differ as little as we please.*

Nothing is stated here as to *how* the variable is to be "made to differ." This defect may be remedied by stating the definition as follows:

2. *The* **limit** *of a (dependent) variable is a constant quantity from which the variable may be made to differ as little as we please by choosing the values of the independent variable sufficiently little different from some quantity called its limit, which is determined arbitrarily or by the conditions of the particular problem under consideration.*

In still other words:

3. *If, as the independent variable approaches near at will to its limit, the dependent variable consequently approaches near at will to some fixed quantity, the latter is called the* **limit** *of the dependent variable, or better, it is called the* **limit** *which the dependent variable approaches as the independent variable approaches the limit fixed for it in the particular discussion in hand.*

These different phrasings of the same idea are given in order that the essential nature of a limit may thereby be made clearer to the student. As working definition, let him adopt that which is most satisfactory to himself; either the second or the third will probably be sufficient for all the cases we shall take up. Still the notion of a limit is of such

fundamental importance, that we give also an exact definition, to which the student may have recourse in case the definition which he has adopted seems no longer quite satisfactory or applicable.

4. Rigorous definition of a limit. *Given an independent variable x and a variable y dependent upon x; and considering two numbers l and k not involving x; if for every positive number, ϵ (no matter how small), a positive number δ_ϵ exists such that $y - l$ is numerically less than ϵ for all values of x $(x \neq k)$, such that $x - k$ is numerically less than δ_ϵ, then y is said to approach the limit l, as x approaches the limit k.*

Remarks. 1. The symbol δ_ϵ is used in this definition to emphasize by the notation that the value of δ is dependent upon that of ϵ.

2. The restriction, $x \neq k$, means simply that the questions as to the existence and the value of the limit may be determined without taking into account the value or the nature of y when $x = k$.

3. The restriction $x \neq k$ is equivalent to the two following restrictions: either $x > k$, or $x < k$. If $x \neq k$ be replaced by $x > k$, we have the definition of the limit which y approaches, as x approaches the limit k through values greater than k; and if $x \neq k$ be replaced by $x < k$, we have the definition of the limit which y approaches, as x approaches k through values less than k. The two limits thus defined are usually the same, but not necessarily so. When this is not the case the function is said to be *discontinuous* for $x = k$, and illustrations will be given when the subject of continuity is taken up (pp. 160–165).

4. It is not necessary that the independent variable be free from all restrictions. It is sufficient that it be free to assume such values as are requisite for the application of the definition. Thus, if $\arcsin x$ be the dependent variable, the independent variable, x, is subject to the restriction that it may not be numerically greater than unity. Likewise, when x grows large without bound, it may often do so through the sequence of positive integral values: for instance, the number of sides of a regular polygon inscribed in a circle may be taken as independent variable; it may grow large without bound, but is always a positive integer.

It often happens that the dependent variable approaches a limit, as the independent variable increases without bound.

To avoid complexity, this alternative has not been included in the above definitions. It is easily seen how the definition of a limit should be modified to include this case. Definition **3**, for instance, would read:

3a. *If as the independent variable increases without bound, the dependent variable consequently approaches near at will to some fixed quantity, the latter is called the* **limit** *which the dependent variable approaches as the independent variable increases without bound.*

ART. **4. Application of the definition; further illustrations.** VI. Let y denote the fraction $\frac{x^2-4}{x-2}$, and let x approach 2. We may write

$$y=(x+2)\frac{x-2}{x-2},$$

whence we see readily that when

$$x = 1,\ 1.5,\ 1.8,\ 1.9,\ 1.99,\ 1.999, \text{ respectively,}$$

then

$$y = 3,\ 3.5,\ 3.8,\ 3.9,\ 3.99,\ 3.999, \text{ respectively.}$$

As the values of x approach 2, those of y approach 4, and y may be made to differ little at will from 4, by taking x near enough to 2. Accordingly y or $\frac{x^2-4}{x-2}$ approaches the limit 4 as x approaches 2.*

* Recurring to the strict definition, we have here $l=4$, $k=2$; if ϵ be $\frac{1}{1{,}000{,}000}$, then $y-4$ will be numerically less than $\frac{1}{1{,}000{,}000}$ provided $x-2$ is numerically less than $\frac{1}{1{,}000{,}000}$; *i.e.* δ_ϵ is $\frac{1}{1{,}000{,}000}$; similarly if ϵ be still smaller, a value δ_ϵ exists such that $y-4$ is numerically less than ϵ, whenever $x-2$ is numerically less than δ_ϵ.

In illustration VI, if we let the independent variable actually reach the limit 2, y assumes the form $\frac{0}{0}$, which may have any value whatever. In all the other illustrations, the value of the dependent variable remains quite clear and unambiguous if the independent variable is made equal to the limit fixed for it. As the illustrations have shown, *this is not a material distinction; the limits are determined according to the same definition and by the same process in each case.*

The application of the definition in the determination of the limit in any specific case does not require the examination of the expression in hand to see what would be its character if the independent variable were put equal to its limit.* This is expressly stated in definition 4, and is understood with the others. Noting, therefore, once for all, that in determining the limit of a dependent variable the independent variable is not to be put equal to its limit, as fixed in the discussion in hand, it will be permissible to perform operations which would not be valid without this proviso; in particular, to divide by a quantity which *would* be zero if the independent variable were equal to its limit.

Illustration VI can now be treated as follows:

Introducing the notation $\lim\limits_{x \doteq a}$ to denote "the limit, as x approaches the limit a, of ⋯," we have

$$\lim_{x \doteq 2} \frac{x^2 - 4}{x - 2} = \lim_{x \doteq 2} \frac{(x+2)(x-2)}{x-2}.$$

---

* When this is done, the function may have a single value, several values, a boundless number of values, or no value (being meaningless); having a single value, this value may or may not be equal to the limit. Instances in which the limit and the value are distinct will be given in discussing continuity, (p. 162).

Since, in accordance with our definition, x is not to be given the value 2 in the determination of this limit, $x - 2$ will not assume the value zero in this discussion, and we may divide numerator and denominator by it, with the result that

$$\lim_{x \doteq 2} \frac{x^2 - 4}{x - 2} = \lim_{x \doteq 2} (x + 2),$$

and the latter limit is seen by inspection to be 4.

The need for the notion of a limit is felt when we have to deal with expressions which (like that in VI) lose definiteness for a certain value of the independent variable. A very common mode of determining the limits of such expressions is to try to transform the expression (as was done in VI) so that it is made up of expressions which would remain unambiguous under these circumstances, and whose limits can accordingly be determined by inspection. This procedure will be repeatedly exemplified in subsequent chapters.

VII.

$$\lim_{x \doteq -2} \frac{x^2 - x - 6}{x^2 + 7x + 10},$$

$$= \lim_{x \doteq -2} \frac{(x - 3)(x + 2)}{(x + 5)(x + 2)},$$

$$= \lim_{x \doteq -2} \frac{x - 3}{x + 5} = -\frac{5}{3}.$$

VIII. To find the limit of $\frac{1}{x}$ as x grows beyond all limits, we notice that by taking x sufficiently large, $\frac{1}{x}$ may be made to differ little at will from zero. Accordingly, $\frac{1}{x}$ approaches the limit zero as x increases without bound.

IX. Similarly, the limit as x grows beyond all bounds, of $\frac{3}{5x^2 + 7x - 5}$, is zero, since by taking x sufficiently large, the value of the fraction may be made small at will.

X. To find the limit of $\frac{2x+1}{5x+3}$ as x grows without bound, we notice that for all values of x (except $x = 0$),

$$\frac{2x+1}{5x+3} = \frac{2+\frac{1}{x}}{5+\frac{3}{x}};$$

and, in the right member, both $\frac{1}{x}$ and $\frac{3}{x}$ approach the limit zero as x increases without bound, and hence $\frac{2}{5}$ is the limit sought.

ART. 5. **Concerning infinity.** When a variable, x, has the property of assuming values which grow larger and larger without bound (Lat. *in-finitus*), we often say, for brevity, that "x becomes **infinite**," or that "x approaches **infinity**" (symbol, ∞). These expressions mean neither more nor less than "x grows large without bound," and this meaning is frequently denoted by the symbol, $x \doteq \infty$, which may be read in any of the above three forms indifferently. The terms *infinite* and *infinity* are always used as abbreviations, and the full meaning of the abbreviation must be clearly understood. Infinity is not a quantity nor a value, though it is sometimes used with the same phraseology as if it were a value. For instance, it is customary to say, $\tan 90° = \infty$, $\log 0 = -\infty$, etc. But though the use of such expressions may add to compactness of form, it must never be forgotten that we are stating a *property*, not a *value*, of the variable in question. This property is that, under certain circumstances, the variable may grow large without bound; the circumstances usually involve some considerations of limits.

EXAMPLES

1. $$\log 0 = -\infty$$

is simply an abbreviation for the statement that when x approaches the limit zero, $\log x$ is negative and grows large numerically without bound.

2. $$\tan 90° = \infty$$

is an abbreviation for: "The tangent of an angle grows large without bound as the angle approaches the limit 90°."

3. *Parallel straight lines meet at infinity,* is merely an abbreviation for the following: "Given a fixed straight line, and a movable straight line intersecting it; if a point P of the movable straight line be kept fixed, and the straight line be turned about this point, then the straight line through P, parallel to the fixed straight line, is the limiting position which the movable straight line approaches, as its point of intersection with the fixed straight line is moved to a distance growing greater without bound."

ART. 6. Further examples of limits.

XI.
$$\lim_{x \doteq 90°} \frac{\tan x}{\sec x} = \lim_{x \doteq 90°} \frac{\sin x}{\cos x} \cdot \cos x$$
$$= \lim_{x \doteq 90°} \sin x$$
$$= 1.$$

XII.
$$\lim_{x \doteq \infty} \frac{x^2 + 3x - 5}{2x^2 - 5x + 7} = \lim_{x \doteq \infty} \frac{1 + \frac{3}{x} - \frac{5}{x^2}}{2 - \frac{5}{x} + \frac{7}{x^2}}$$
$$= \frac{1}{2},$$

since each of the fractions in the numerator and the denominator approaches zero when x increases without bound.

XIII.
$$\lim_{x \doteq \infty} \frac{2x^3 - 4x^2 + 9x}{5x^2 - 6x + 2} = \lim_{x \doteq \infty} \frac{2x - 4 + \frac{9}{x}}{5 - \frac{6}{x} + \frac{2}{x^2}}.$$

As x grows large without bound, the denominator approaches 5, while the numerator grows large without bound, and hence the whole fraction grows large without bound. In our abbreviated form we may state this as follows:

$$\lim_{x \doteq \infty} \frac{2x^3 - 4x^2 + 9x}{5x^2 - 6x + 2} = \infty.$$

XIV. Sometimes it is advantageous to pass to logarithms as the first step in the determination of the limit. Thus, to find $\lim_{n \doteq \infty} \sqrt[n]{3}$, we put

$$y = \sqrt[n]{3},$$

and have
$$\log y = \frac{1}{n} \log 3,\text{*}$$

$$\lim_{n \doteq \infty} \log y = \lim_{n \doteq \infty} \frac{\log 3}{n}.$$

$$= 0.$$

Hence,
$$\lim_{n = \infty} y = 1.\dagger$$

Art. 7. The fundamental theorem of limits. The idea of limits is made useful and available in mathematical investigations by the following fundamental theorem:

If two variable quantities are always equal and each approaches a limit, those limits are also equal.‡

This theorem is almost self-evident when we understand clearly the meaning of the expressions employed in it. If two quantities are *always* equal, they are *identical;* how-

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* Formula 9, Appendix. † Formula 8, Appendix.

‡ Of course, we can speak of equality between two expressions only when both have an unambiguous meaning. A fuller and equivalent wording of the above theorem would be: *If two variable quantities are always equal whenever each has a definite meaning, and if, while varying simultaneously, each approaches a limit, those limits are also equal.*

ever the expressions may vary, they have always the same value; they can be only different forms of expression for the same thing. Whatever can be said about one can be said (with the proper change in *form* merely) about the other.

For instance, if $x = z^2$, the equation

$$x^2 - 16 = z^4 - 16$$

is an identity; it is true for all values of z and x, the latter being fixed (by the relation $x = z^2$) as soon as z is fixed. If we discover that $x - 4$ is a factor of the left member, it follows without further investigation that $x - 4$, expressed in terms of z (*i.e.* $z^2 - 4$), is a factor of the right member.

In particular, if the variable expressed in one form approaches a limit, the same variable, expressed in another form, will approach the same limit expressed in a corresponding form. This, then, is what the theorem means:

If we have two expressions for the same variable quantity, and if simultaneously under certain circumstances each of these expressions approaches a limit, these limits can be simply two different expressions for the same thing.

As an illustration, let us take the identity already used,

$$x^2 - 16 = z^4 - 16; \quad (x = z^2).$$

If x approaches the limit 4, we see that the left member approaches zero. We know then, from this fact alone, that the right member (being only another form of expression for the left member), approaches zero also when x approaches 4, or, in terms of z, when z^2 approaches 4, or when z approaches 2 (or -2). Here we find the limit of the right member by expressing the limit of the left member in the notation of the right member. But we might just as well have determined independently the limit which the right member approaches when z approaches ± 2 (which is equivalent to

x approaching 4). We know in advance that the results must be equal, being the limits, under the same circumstances, of different expressions for the same thing. The equality between these limits may be a relation of interest. For instance, suppose that we had proved somehow that

$$x^2 - a = z^4 - b$$

for all values of x and of z subject to the condition $x = z^2$; then, as x approaches zero, the left member approaches $-a$, and at the same time the right member approaches $-b$. We have, therefore, $-a = -b$, or $a = b$. This is a new relation between the quantities a and b which may be of value.

As another illustration, consider the area of a regular polygon inscribed in a circle; it is $\frac{AP}{2}$, where A denotes the apothegm, and P the perimeter of the polygon. Calling the area of the polygon S, we have S and $\frac{AP}{2}$ as two different expressions for the area of the polygon. But as the number of sides is increased, S approaches the circle as its limit, and $\frac{AP}{2}$ approaches the limit $\frac{r \cdot 2\pi r}{2}$, or πr^2 (r denoting the radius of the circle). These are two different expressions for the same thing (the limit of the area of the polygons); therefore

$$\text{Area of circle} = \pi r^2.$$

This fundamental theorem adds one to the ways in which we can deduce a new equation from one already known. We are able to deduce new equations from given equations by various methods, such as adding the same quantity to both members; multiplying both by the same factor; raising both to the same power; and the like. In all these

cases we know that the resulting equation will hold true whenever the original equation does so. We can now add to these another method for deducing a new equation, *viz.* by equating the limits of the two members of the original equation. This method is subject to the important proviso, that the equation from which we start must hold true for all values of the variable quantity or quantities involved, — must be an *identity*. The other methods mentioned did not labor under this restriction.

An identical equation having been established, a new identical equation can be deduced from it by equating the limits of both members.

We say this new equation is deduced by "taking the limit of the given equation," or by "passing to the limit." The resulting equation is just as accurate and as rigorously deduced as that found, for instance, by squaring both members. This is true because: *The limit of a variable (whenever any exists) is a precise quantity and independent of the variable.* It is *not* an approximation, but the exact quantity to which, under certain circumstances, the variable approximates.*

This method of deducing new equations is fundamental to the applications of our subject.

ART. 8. **Propositions concerning limits.** There are certain propositions concerning limits, one or more of which must be implied in almost every case of the determination of a limit. They are quite plausible to beginners, who

* In some cases, variables may actually become equal to their limits, in others not; but in *all* cases, the variable may approximate closely at will to the limit. We shall see later that this property may be utilized to determine, with any desired degree of approximation, the numerical value of quantities proved (or defined) to be the limits of certain variables.

usually tacitly assume their truth, and apply them without having ever consciously formulated them.

They are the following:

I. The limit of the sum of a fixed number of terms is the sum of the limits of the terms considered separately.

II. The limit of the product of two factors is the product of the limits of the factors considered separately.*

III. The limit of a fraction is the limit of the numerator divided by the limit of the denominator. †

Proofs of these propositions will be given in Art. 11, p. 92.

ART. 9. **Concerning epsilons.** Quantities which can be made small at will, and which are such functions of the independent variable that they do, in fact, approach zero when the independent variable approaches the limit which may be selected for it in the problem under consideration, are often denoted by the Greek letter ϵ, which is read "epsilon." *An* **epsilon** *is a quantity which approaches zero under the conditions of the discussion in which it occurs.* If various epsilons occur in the same discussion, they may be distinguished by subscripts, as $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \cdots, \epsilon_n$.

We can express the statement that l is the limit of x ‡ (in the form of an equation) by use of an ϵ, *viz.*

$$x - l = \epsilon.$$

* There is one exception, *viz.* the case in which one factor approaches zero while, at the same time, the other grows boundlessly large.

† The exceptional cases, in which the limit of the denominator is zero, will be considered in connection with the proof of the proposition.

‡ Such statements as "l is the limit of x," which we shall often employ for brevity, mean that the variable x approaches the limit l when the independent variable approaches a certain limit, fixed for it by the conditions of each particular problem.

Conversely, $y - k = \epsilon_1$,

expressed in words, is nothing other than the statement that the limit of y is k. The two forms of statement are quite equivalent.

Art. **10. Properties of epsilons.** 1. *The sum of a fixed number of epsilons is an epsilon.* To show this we have to show that this sum can be made small at will. Let the number of epsilons be n. Then, however small the sum may be desired to be, it can be made so by taking each of the constituent epsilons smaller than $\frac{1}{n}$th of the desired sum. That is, the sum in question can be made small at will; it is, therefore, by definition, an epsilon. This result may be expressed in an equation as

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n = \epsilon.$$

2. *The product of a constant, c, and an epsilon, ϵ, is an epsilon.* For however small the product is to be made, it can be made so, by taking ϵ smaller than $\frac{1}{c}$th of the desired value. The product can be made small at will, and is hence an epsilon.

3. *The product of any number of epsilons is likewise an epsilon.* For when each factor can be made small at will, the whole product can be made small at will.

Art. **11. Proof of the propositions concerning limits.** I. *The limit of the sum of a fixed number of terms is the sum of the limits of the terms considered separately.*

Let $x_1, x_2, x_3, \cdots, x_n$ be the terms, and $l_1, l_2, \cdots, l_n$ the limits which they respectively (and simultaneously) approach under the conditions of the problem.

Then we have (cf. p. 91)

$$
(1) \qquad
\begin{aligned}
x_1 - l_1 &= \epsilon_1, \\
x_2 - l_2 &= \epsilon_2, \\
x_3 - l_3 &= \epsilon_3, \\
\cdots \quad &\cdots \quad \cdots \\
x_n - l_n &= \epsilon_n.
\end{aligned}
$$

We wish to show that

$$(x_1 + x_2 + x_3 + \cdots + x_n) - (l_1 + l_2 + l_3 + \cdots + l_n)$$

is an epsilon.

Adding the equations (1), we have

$$\begin{aligned}(x_1 + x_2 + x_3 + \cdots + x_n) - (l_1 + l_2 + l_3 + \cdots + l_n)& \\ = \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n.&\end{aligned}$$

By the first of the properties of epsilons proved above, the right member is an epsilon; which was to be shown.

II. *The limit of the product of two factors is the product of the limits of the factors considered separately.*

Let the two variables be x and y, and l and m their respective limits. Then we wish to show that lm is the limit of xy. The hypothesis is

$$x - l = \epsilon_1, \quad y - m = \epsilon_2,$$

and we wish to show that

$$xy - lm = \epsilon.$$

We have $\qquad x = l + \epsilon_1, \quad y = m + \epsilon_2,$

and hence $\qquad xy = lm + l\epsilon_1 + m\epsilon_2 + \epsilon_1\epsilon_2,$

or $\qquad xy - lm = l\epsilon_1 + m\epsilon_2 + \epsilon_1\epsilon_2.$

The terms of the right member can each be made small at will, hence their sum can be made small at will, and the right member is an epsilon; accordingly,

$$xy - lm = \epsilon.$$

III. *The limit of a fraction is the limit of the numerator divided by the limit of the denominator.*

Let x and y approach simultaneously the limits l and m respectively. Then we wish to show that $\frac{x}{y}$ approaches $\frac{l}{m}$.

We have $x - l = \epsilon_1$ and $y - m = \epsilon_2$, and we wish to show that

$$\frac{x}{y} - \frac{l}{m} = \epsilon.$$

$$\begin{aligned} \frac{x}{y} - \frac{l}{m} &= \frac{l + \epsilon_1}{m + \epsilon_2} - \frac{l}{m} \\ &= \frac{m(l + \epsilon_1) - l(m + \epsilon_2)}{m(m + \epsilon_2)} \\ &= \frac{m\epsilon_1 - l\epsilon_2}{m(m + \epsilon_2)}. \end{aligned}$$

In the last fraction the numerator can be made small at will while the denominator approaches m^2. If m is not zero, the fraction can therefore be made small at will; accordingly,

$$\frac{x}{y} - \frac{l}{m} = \epsilon.$$

Exceptional cases. (1) In case m is zero and l is not, then in the fraction $\frac{x}{y}$, the denominator grows small at will, while the numerator does not; that is, the fraction grows large at will.

(2) In case l and m are both zero, we cannot tell immediately what limit the fraction approaches, but must first transform the fraction in some suitable manner before determining the limit; as was done, for example, in illustrations VI and VII above.

EXERCISES X

Find the limits indicated in the following expressions:

1. $\lim\limits_{x \doteq 4} \dfrac{x^2 + 2x - 24}{x^2 - 7x + 12}$.

2. $\lim\limits_{x \doteq 0} \dfrac{3x^2 - 5x}{2x^3 - 15x}$.

3. $\lim\limits_{k \doteq 0} \dfrac{(x + k)^2 - x^2}{k}$.

4. $\lim\limits_{r \doteq 0} \dfrac{(x^2 - 2r)^3 - x^6}{6r}$.

5. $\lim\limits_{x \doteq 0} \dfrac{x^{12} - 3x^{11} + x^7}{2x^{14} - 3x^{11} + 5x^5}$.

6. $\lim\limits_{x \doteq 0} \dfrac{x^4 - 2x^2 + 3x}{4x^3 - 6x}$.

7. $\lim\limits_{x \doteq r} \dfrac{ax^2 - 2arx + ar^2}{bx^2 - 2brx + br^2}$.

8. $\lim\limits_{x \doteq 1} \dfrac{1 - x^3}{1 - x}$.

9. $\lim\limits_{h \doteq 0} \dfrac{(x + h)^4 - x^4}{h}$.

10. $\lim\limits_{x \doteq c} \dfrac{x^5 - c^5}{x - c}$.

11. $\lim\limits_{x \doteq 1} \dfrac{x^4 - 1}{x - 1}$.

12. $\lim\limits_{x \doteq q} \dfrac{x^n - q^n}{x - q}$. *Ans.* nq^{n-1}.

13. $\lim\limits_{n \doteq \infty} \dfrac{1}{n^2}$.

14. $\lim\limits_{r \doteq \infty} \dfrac{r}{r + 1}$.

15. $\lim\limits_{n \doteq \infty} \dfrac{n(n + 1)}{(n + 2)(n + 3)}$.

16. $\lim\limits_{x \doteq \infty} \dfrac{3x^2 - 5}{2x^2 - 6x}$.

17. $\lim\limits_{n \doteq \infty} \dfrac{(n+1)(n+2)(n+3)}{5n^3}$.

18. $\lim\limits_{n \doteq \infty} \left(\dfrac{1}{2}\right)^n$.

19. $\lim\limits_{n \doteq \infty} \left(\dfrac{2}{3}\right)^n$.

20. $\lim\limits_{n \doteq \infty} \dfrac{n^4}{(n+1)(2n-1)(1-3n)}$.

21. $\lim\limits_{x \doteq 0} \dfrac{x^6 - 5x^5 + 2x^2}{x^8 - 3x^3 - x^2}$.

22. $\lim\limits_{x \doteq -1} \dfrac{x^2 + 4x + 3}{x^2 - 7x - 8}$.

23. $\lim\limits_{x \doteq 0} \dfrac{a - \sqrt{a^2 - x^2}}{x^2}$. *Ans.* $\dfrac{1}{2a}$.

HINT. Rationalize the numerator.

24. $\lim\limits_{x \doteq 1} \dfrac{x^7 - 6x^6 + x}{(1 - x)^2}$. *Ans.* 15.

HINT. Make use of the theorem that if a polynomial vanishes when a is substituted for x, then $x - a$ is a factor of the polynomial. The other factor may be found by actual division.

25. $\lim\limits_{x \doteq 2} \dfrac{x^2 - 4}{\sqrt{x + 2} - \sqrt{3x - 2}}$. *Ans.* -8.

HINT. Put $x = y + 2$, and find the limit of the result when $y \doteq 0$ After substitution, rationalize the denominator.

26. $\lim\limits_{x \doteq -1} \dfrac{x^5 + 3x^4 - 5x^3 - 7x^2 + 3x + 3}{x^6 + x^5 - 4x^3 + 5x^2 + 8x - 1}$. *Ans.* $\frac{1}{3}$.

HINT. Either the method indicated for **24** or that for **25** may be used.

CHAPTER III

THE FUNDAMENTAL CONCEPTIONS OF THE DIFFERENTIAL CALCULUS

ART. 1. **The underlying principles.** The Calculus has for its subject of study, **continuous** quantity, *i.e.* quantity which varies without a break from one value to another. Time and motion are illustrations of continuous variation. Indeed, the phenomena of nature are generally of this character. When a planet under the influence of a perpetually varying force revolves around the sun; when the air, in propagating sound, occasions by its vibrations ever-changing states of rarefaction and condensation; when by the explosion of a mixture of hydrogen and of oxygen gas the temperature rises very rapidly to a maximum only to fall nearly as rapidly, we are always dealing with phenomena that are varying continuously. Consequently, the careful study of any aspect of nature soon requires the application of the Calculus. When Leibnitz * and Newton † laid the

* Gottfried Wilhelm Leibnitz (1646–1716) was a man of many-sided genius who left a permanent impress upon Philosophy, Theology, Philology, Geology, and other subjects, as well as upon Mathematics. His presentation of the Calculus appeared in a paper entitled: "*Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus*," published in the *Acta Eruditorum*, Leipzig, 1684.

† Sir Isaac Newton (1642–1727) made his principal publications on our subject, in the two following works: *Philosophiæ naturalis principia mathematica*, published in 1687, and *Methodus fluxionum et serierum infinitarum, cum ejusdem applicatione ad curvarum geometriam*, first published in 1736 (in an English translation), but said to have been finished in 1671.

foundations of the Differential Calculus, in all probability independently, they did not perhaps fully realize that an aid to the investigation of the problems, whether of pure mathematics or of nature, second to none in power and fertility, would be evolved from their ideas. But in the two centuries that have since elapsed, these ideas have not only given rise to a large system of results of the greatest importance in mathematics, but they have also been applied more and more in the various branches of science, and have extended over the entire realm of physical phenomena in so far as we have been able to subject them to measurement.

To develop an outline of these far-reaching methods, and to show also how the problems of mathematics and the phenomena of nature may be treated by their means, is the chief object of this book. These methods are characterized by certain unique ideas and notions of fundamental importance. There seems to be a widespread opinion that they are very difficult to understand; but we take occasion to remark with emphasis that this is not the case. With precise formulations, the difficulties vanish almost entirely; wherever they may still occur, they are due not so much to the notions and methods of our subject itself, as to the nature of the problems or the phenomena to which they are applied. The mathematical portion of the discussion requires nothing more than the same careful formulation of data and hypotheses, the same precautions in drawing conclusions, as other branches of mathematics, and its results are equally accurate.

We begin by discussing several problems whose solution requires the application of the underlying principles of the Differential Calculus.

ART. 2. **Motion on the parabola.** *Given that a point moves on a parabola; to calculate the direction of its motion at any instant.*

The direction of motion changes at every moment, but it can be represented at any position in its path by the direction of the tangent to the parabola at that point. If we had the figure of the parabola before us, we could determine the direction of its tangent at any point by actual measurement; but the problem which we have to solve requires us to obtain for the direction a formula which is true for all points.

For this purpose we consider the parabola (Fig. 35) to have the y-axis as the axis of symmetry. Its equation (interchanging x and y in the equation deduced on p. 21) will assume the form

$$(1)\quad x^2 = 2py \text{ or } y = \frac{x^2}{2p}.$$

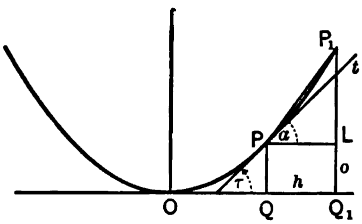

FIG. 35.

Let the point P, for which the position of the tangent is to be calculated, have the coördinates x, y. Let the tangent at P be t and the angle which it makes with the axis of x be τ. We have now to determine this angle. We can easily reach an approximate result by substituting for the parabola an inscribed polygon with a very large number of sides, and determining the direction of the side PP_1 passing through P. If P_1 has the coördinates $x_1 y_1$, and if α be the angle which the side PP_1 makes with the axis of X, it follows from the right-angled triangle PP_1L that

$$(2)\qquad \tan\alpha = \frac{P_1L}{PL} = \frac{P_1Q_1 - LQ_1}{QQ_1} = \frac{y_1 - y}{x_1 - x}.$$

Since P and P_1 are points of the parabola, their coördinates satisfy respectively the equations

$$y = \frac{x^2}{2p}, \text{ and } y_1 = \frac{x_1^2}{2p},$$

and by subtraction

$$y_1 - y = \frac{x_1^2 - x^2}{2p};$$

when we substitute this value in (2) we obtain

$$\tan\alpha = \frac{1}{2p}\,\frac{x_1^2 - x^2}{x_1 - x}. \tag{3}$$

If we denote the distance QQ_1 by h, so that

$$x_1 - x = h \text{ and } x_1 = x + h, \tag{4}$$

then equation (3) becomes

$$\tan\alpha = \frac{1}{2p}\,\frac{(x+h)^2 - x^2}{h}$$

$$= \frac{1}{2p}\,\frac{2hx + h^2}{h} = \frac{1}{2p}(2x + h),$$

or, finally,

$$\tan\alpha = \frac{x}{p} + \frac{h}{2p}. \tag{5}$$

We have thus determined the direction of the side PP_1, and approximately, the direction of the tangent. The error which we commit depends upon how near the point P_1 lies to P; that is to say, it depends upon the magnitude of h. It is clear that we can take the sides of the polygon which we have substituted for the parabola so small that our eye cannot detect a difference between the figure of the polygon and that of the parabola; and even so small that our most

powerful instruments of measurement cannot enable us to detect any difference between the tangents of the parabola and the sides of the polygon.

When an error is so small that we can neither see nor measure it, it is for all *practical purposes not present;* from the *practical* point of view we have therefore solved the proposed problem. But mathematics demands *perfect* accuracy, and a simple consideration will now enable us to make such use of the foregoing method that it will give us the *absolutely exact value* of the tangent of the angle. We first observe that the right side of equation (5) consists of two terms, of which the first does not contain the quantity h. If we now substitute for h a series of values, as, for example, 0.1 mm., 0.2 mm., etc., the first term is not changed at all; only the second term, which measures the degree of approximation, is changed. If we substitute for h smaller and smaller numbers, as, for example, 0.0000001 mm., the polygon will approach nearer and nearer to the parabola. Of course, we must always distinguish between the polygon and the parabola, no matter how small h becomes. We can never, in our thoughts, bring the polygon to coincidence with the parabola, but our mathematical methods enable us to deduce the equation which must hold true for the parabola from that which we know holds true for the polygon; all that we have to do is to consider the limits which both members approach as h approaches zero. These values are equal by the fundamental theorem of limits (p. 87), so that we have

$$\tan \tau = \frac{x}{p}.$$

We have, accordingly, thus obtained the actual value of the tangent of the angle τ, and this equation represents the

direction of the tangent to the parabola at the point P, and since the point P is *any* point of the parabola, it represents the direction of the tangent at *every* point with *perfect accuracy*.

ART. 3. **Concerning speed.** The processes of nature may take place uniformly or with varying speed. We can form no clear conception of the latter, and find it necessary to express it somehow as uniform change.

If a body moves uniformly, we define its **speed** as the ratio of the distance traversed to the time taken. But if it moves with varying speed, we assume, in order to get an idea of its speed at any given moment, that at that moment it moves *uniformly* for the brief interval of time τ, and in this time traverses the distance σ; the ratio of the distance σ to the interval of time τ gives the **mean speed** with which the body moves over this distance. When a body has a varying motion, we are accordingly accustomed to define the **speed at any moment** to be the speed which the body would have, if, at the moment under consideration, it moved on uniformly. Such a procedure is strictly necessary, for, as has just been said, our conception of speed is limited to that of uniform speed. When *direction* is an essential element of any motion, the ratio defined above as speed is called **velocity.**

In general, we define the **speed** of any change in nature to be the ratio of the amount of this change to the time taken, with analogous definitions for *mean speed* and *speed at any moment*.

ART. 4. **The motion of a freely falling body.** *To determine the velocity at any instant, of a freely falling body.*

When a body falls vertically downward from a state of rest, we know that the distances traversed in 1, 2, 3, 4, ⋯

seconds are $\frac{g}{2}$, $4\frac{g}{2}$, $9\frac{g}{2}$, $16\frac{g}{2}$, $\cdots$ units of length (g being the velocity at the end of the first second of fall), and that in general the distance l gone over in t seconds may be expressed by the formula

$$(1) \qquad l = \tfrac{1}{2}gt^2.$$

The velocity of the motion at different instants is different, for the distances traversed during any given second have lengths equal to $\frac{g}{2}$, $3\frac{g}{2}$, $5\frac{g}{2}$, $\cdots$, increasing continually with the time. But, as we have already stated above (p. 102), our conception of velocity is limited to cases wherein *equal* distances are passed over in equal intervals of time. Thus we are again confronted by the difficulty that the conceptions which we have to employ in our operations are not directly applicable to the phenomenon as it actually occurs, and hence we must have recourse to a method of approximation.

To simplify matters, we substitute for the falling body a point having weight, for instance, the weight of the body concentrated at its center of gravity. Let P_0 (Fig. 36) represent the place where the motion begins, and let the falling point reach the positions P, P_1, P_2, $\cdots$, in t, t_1, t_2 $\cdots$ seconds, and let l, l_1, l_2, $\cdots$ stand for the distances $\overline{P_0P}$, $\overline{P_0P_1}$, $\overline{P_0P_2}$ $\cdots$ traversed. According to (1), we have the equations

$$(2) \qquad l = \tfrac{1}{2}gt^2,\ l_1 = \tfrac{1}{2}gt_1^2,\ l_2 = \tfrac{1}{2}gt_2^2 \cdots.$$

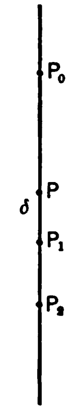

FIG. 36.

We now imagine that there is a second point also moving vertically downward from P_0, which passes the positions P, P_1, P_2, P_3, $\cdots$ at the same instants as the first point, but

traverses the distances $\overline{PP_1}$, $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\cdots$ with velocities which are *uniform* throughout each distance. Both of the points will have different motions throughout these distances; we shall see them at any instant in different positions within these distances, although they pass the positions P, P_1, P_2, $\cdots$ at the same instants.

If δ is the length of $\overline{PP_1}$, and τ is the time it takes for the second point to pass over this distance, its velocity is

$$(3) \qquad V = \frac{\delta}{\tau}.$$

Now $\qquad \delta = PP_1 = P_0P_1 - P_0P = l_1 - l,$

and, since the positions P and P_1 are passed at the end of t and t_1 seconds,

$$(4) \qquad \tau = t_1 - t;$$

(5) therefore, $\qquad V = \dfrac{\delta}{\tau} = \dfrac{l_1 - l}{t_1 - t}.$

But according to (2),

$$l_1 - l = \tfrac{1}{2} g (t_1^2 - t^2),$$

and by substituting this value of $l_1 - l$ in (5), we have

$$V = \tfrac{1}{2} g \frac{t_1^2 - t^2}{t_1 - t}.$$

By (4) this becomes

$$V = \tfrac{1}{2} g \frac{(t + \tau)^2 - t^2}{\tau},$$

or, reducing,

$$(6) \qquad V = gt + \frac{g}{2}\tau;$$

this is the velocity of the second point throughout the entire distance PP_1.

We can make the motion of the second point approximate as closely as we wish to the motion of the freely falling point. The degree of approximation depends upon how small the distances PP_1, P_1P_2, $\cdots$ are taken.

But we cannot conceive of the perfect coincidence of the motion of the second point with that of the first. It is just as difficult for us to form a conception of such a coincidence as it is to conceive of the transition of a polygon into a parabola; still *the method of limits helps us out again;* if we determine the limit which the velocity of the auxiliary point approaches as τ approaches zero, that limit is the *exact* value of the velocity v at the time t. We find thus

$$v = gt, \tag{7}$$

and, since P is an arbitrary point, this formula defines the velocity of a freely falling body at every moment of its motion.

ART. 5. **The linear expansion of a rod.** *To ascertain how a rod expands at any moment while being heated.* Experiment shows that a rod whose length at the temperature of melting ice we may call unity, on being heated expands in such a way that its length l at any temperature θ can be represented by the expression

$$l = 1 + b\theta + c\theta^2, \tag{1}$$

in which b and c stand for constant numbers that depend upon the nature of the rod, and can be determined experimentally.

We define first, the **coefficient of expansion** at any temperature θ, as the increase in length which the rod would undergo during a rise of one degree in temperature above θ, if the

expansion continued uniformly in this interval. We seek to determine the coefficient of expansion of our rod.

Let l, l_1, $l_2 \cdots$ be the lengths of the rod corresponding to the temperatures θ, θ_1, $\theta_2 \cdots$. We also suppose that the expansion occurs *uniformly* when the rod is heated from θ to θ_1, from θ_1 to θ_2, etc. The rod accordingly increases uniformly in length from l to l_1 when it is warmed from θ to θ_1; hence, the expansion A, which corresponds to a rise in temperature of one degree, is

$$A = \frac{l_1 - l}{\theta_1 - \theta}. \tag{2}$$

According to (1)

$$l_1 = 1 + b\theta_1 + c\theta_1^2,$$

$$l = 1 + b\theta + c\theta^2,$$

and by subtraction

$$l_1 - l = b(\theta_1 - \theta) + c(\theta_1^2 - \theta^2).$$

If we put

$$l_1 - l = \lambda, \quad \theta_1 - \theta = \Theta, \tag{3}$$

so that λ is the increase of length corresponding to a rise of temperature equal to Θ, and bear in mind that

$$\theta_1^2 - \theta^2 = (\theta_1 + \theta)(\theta_1 - \theta) = \Theta(\theta_1 + \theta),$$

we have

$$A = \frac{\lambda}{\Theta} = \frac{b\Theta + c\Theta(\theta_1 + \theta)}{\Theta},$$

or finally, after substitution of $\theta + \Theta$ for θ_1

$$A = b + c(2\theta + \Theta) = b + 2c\theta + c\Theta. \tag{4}$$

This is the coefficient of expansion for the difference of temperature $(\theta_1 - \theta)$, it being assumed that the expansion is uniform.

The smaller we make Θ (the difference between θ_1 and

θ), the nearer our representation of the process of expansion approximates to its actual course at the moment when the temperature is θ. While here, as in the previous instances, a complete coincidence cannot be conceived, the formula gives the correct result. We determine the limit which A approaches as Θ approaches zero, and thus get for the required coefficient of expansion α at the moment the temperature is equal to θ

$$(5) \qquad \alpha = b + 2c\theta;$$

since θ may have any value, this formula holds for the entire course of the process.

It is worth while to remark that the coefficient of expansion can also be defined as a speed. It is the measure of the rapidity with which the increase of length takes place when the heating occurs uniformly. We have just as clear an idea of the rapidity of a process as we have of the rapidity of a motion. The increase of length per degree accompanying a uniform rise of temperature corresponds perfectly to the increase in distance traversed by a moving body per second; in accordance with this the quantity α may be termed the **speed of expansion** at the temperature θ.

ART. 6. **The derivative.** The determination of the tangent of the parabola was based upon the equation

$$(1) \qquad \tan\alpha = \frac{y_1 - y}{x_1 - x} = \frac{x}{p} + \frac{h}{2p},$$

from which by taking the limits of both members we obtained

$$(2) \qquad \tan\tau = \frac{x}{p}.$$

This corresponded to our making the polygon approach the parabola by diminishing the lengths of its sides. There is another notation which is much used and which is sometimes

more convenient. In the problem of motion on the parabola we now denote the difference of the abscissæ by Δx instead of by h and the difference of ordinates by Δy; that is, we put

(3) $$x_1 - x = \Delta x \text{ and } y_1 - y = \Delta y,$$

so that $$x_1 = x + \Delta x \text{ and } y_1 = y + \Delta y.$$

Hence Δx and Δy signify the increments which x and y receive, respectively, when we pass from the point P to the point P_1, and

(4) $$\tan\alpha = \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x};$$

the *limit* of this quotient, as $\Delta x \doteq 0$, is the value of $\tan\tau$; it is called the **derivative** (and also *differential coefficient*), or, more exactly, the **derivative of y with respect to x**, and is represented by $\frac{dy}{dx}$. We have therefore for the parabola

(5) $$\tan\tau = \frac{dy}{dx} = \frac{x}{p}.$$

The case of freely falling bodies is entirely analogous. We started from the equation

(6) $$V = \frac{l_1 - l}{t_1 - t} = gt + \frac{g}{2}\tau.$$

In this case we put

(7) $$t_1 - t = \Delta t \text{ and } l_1 - l = \Delta l,$$

so that Δl indicates the increment of distance for the increment of time Δt; then the limit of the quotient

(8) $$V = \frac{l_1 - l}{t_1 - t} = \frac{\Delta l}{\Delta t},$$

when $\Delta t \doteq 0$, yields the value of v.

As in the previous case, we call the limit which $\frac{\Delta l}{\Delta t}$ approaches, the derivative; or, more accurately, the derivative of l with respect to t, and write for it, $\frac{dl}{dt}$; so that for freely falling bodies the equation holds

$$(9) \qquad v = \frac{dl}{dt} = gt.$$

And, finally, in the last example which we discussed above, we began with the equation

$$(10) \qquad A = \frac{l_1 - l}{\theta_1 - \theta} = b + 2c\theta + c\Theta,$$

and found that the limit of the quotient as Θ approaches zero was an expression giving the coefficient of expansion, *viz.*

$$(11) \qquad \alpha = b + 2c\theta.$$

Here again we put

$$(12) \qquad l_1 - l = \Delta l, \ \theta_1 - \theta = \Delta\theta,$$

so that Δl indicates the increment of length corresponding to the increment of temperature $\Delta\theta$, whence

$$(13) \qquad A = \frac{\Delta l}{\Delta\theta}.$$

As in the previous cases, we call the limit of this quotient when $\Delta\theta \doteq 0$ the *derivative* of l with respect to θ, and denote it by $\frac{dl}{d\theta}$, so that,

$$(14) \qquad \alpha = b + 2c\theta = \frac{dl}{d\theta}.$$

ART. 7. **The physical signification of derivatives.** The foregoing examples may serve to show how various the

problems are to which derivatives may be applied; we may even say that students of science often make use of derivatives unwittingly. Thus, as the derivative of a distance with respect to the time in which it is traversed, expresses the *speed* with which the given distance is traversed, so the derivative of the amount of a substance reacting chemically, with respect to the time, expresses the *speed of reaction.* If we are considering the relationship between the temperature and the volume of a liquid, or the length of a rod, or the electromotive force of a voltaic cell, the derivatives of these quantities with respect to the temperature are their *temperature coefficients.* If a metal, as iron, be subjected to the action of a magnetic field, the metal itself becomes magnetized; that is, it acquires a certain magnetic moment. The derivative of this moment with respect to the intensity of magnetization is called the capacity for magnetization of the metal in question, and characterizes its magnetic behavior.

ART. 8. **The function-concept.** When the pressure to which a gas is subjected is altered, the volume occupied by the gas also changes, expanding or contracting according as the pressure diminishes or increases. The relative change of pressure and volume takes place in accordance with Boyle's Law (p. 3), and the interdependence between pressure and volume comes under the concept, which is known in mathematics as the **function-concept**, and is defined as follows:

The quantity y is a **function** *of the quantity x, if x and y are so related that to every value which x may assume there correspond one or more values of y.*

Hence we speak of the volume of a gas as being a *function*

of the pressure to which it is subjected. Similarly, we speak of the solubility of a substance as being a function of the temperature, and the diameter of a soap bubble as being a function of the pressure of the air within it. Likewise, the law of freely falling bodies expresses a relationship between the distance traversed and the time in which it is traversed, and therefore the distance is a function of the time.

Boyle's Law may be expressed by the equation

$$(1) \qquad vp = v_0p_0, \text{ or } v = \frac{v_0p_0}{p},$$

where v_0 and p_0 are the values of the volume and pressure of the gas in its initial state.

Similarly, for bodies falling from rest, we have the equation

$$(2) \qquad l = \tfrac{1}{2}gt^2,$$

where l is the distance traversed in the time t, and g is a constant.

These equations enable us to calculate for every value of p the corresponding value of v, and for every value of t the corresponding value of l; accordingly v is a function of p, and l is a function of t.

But we have only to put the above equations into the forms

$$(3) \qquad p = \frac{p_0v_0}{v} \text{ and } t = \sqrt{\frac{2l}{g}},$$

to recognize that the pressure p is also a function of the volume v, and t is a function of l; for from these equations we can calculate the values of p and t corresponding to any values assigned to v and l. Which of these two forms of expression should be selected depends upon the problem

with which we are dealing, and the form of the result we seek. In the first case, p and t were regarded as independent variables, and v and l respectively as dependent variables (pp. 77–78), while in the second case, we chose to regard v and l as the independent variables, and consequently p and t as variables dependent upon them. In the processes of nature that require time for their completion, it is customary to regard the time t as the independent variable, since we feel that the time passes in a constantly uniform way which is entirely independent of ourselves, and may therefore well be regarded as a "natural" independent variable. But nothing prevents us from choosing t as the dependent variable for the purposes of calculation, just as was actually done in equation (3); we can, for example, take up the problem: to determine the time t required by a falling point to traverse a given distance.

Another illustration may be taken from Analytic Geometry. In every equation, between the coördinates x and y, which represents a curve, x and y are variable quantities; they can assume a countless number of sets of corresponding values, and their changes are regulated by the law expressed algebraically by the equation in question (and graphically, in the curve corresponding to it).

If x be taken as the independent variable, then y is the dependent variable; by means of the given equation, a value of y can be found corresponding to every value of x, and therefore y is a function of x. But, on the other hand, with the aid of the same equation we can calculate for any value of y the corresponding value of x, that is to say, we can also regard x as a function of y, or consider y as the independent, and x the dependent variable. These sets of values enable us to plot the curve, which is the same whether we determine the ordinate corresponding to each abscissa, or *vice versa*.

We have already become familiar in elementary mathematics with the simplest functions, such as powers, logarithms, trigonometric functions:

$$x^n,\ \log x,\ \sin x,\ \cos x,\ \tan x,\ \cot x,\ \text{etc.};$$

by combining these we can obtain a large number of new functions, as, for example,

$$\frac{a}{x},\ \sqrt{1+x^2},\ \log\frac{a-x}{a+x},\ \sin x+\cos x,\ \text{etc.}$$

As symbols for functions of x, the signs

$$f(x),\ \phi(x),\ F(x),\ L(x),$$

and the like are in general use, and others may be introduced as occasion demands. Thus the equations

$$(4) \qquad y=f(x),\ s=\phi(t),\ w=L(u),\ \text{etc.},$$

mean that y is some function of x, s is some function of t, w is some function of u, etc. If, then, x_1y_1, x_2y_2, x_3y_3, etc., are corresponding values of x and y, this is expressed by the equations

$$(5) \qquad y_1=f(x_1),\ y_2=f(x_2),\ y_3=f(x_3),\ \text{etc.},$$

a mode of expression with which we have become familiar in Analytic Geometry.

As examples of functions taken from nature, we mention the following: The tension of a vapor is a function of the temperature; the time of vibration of a pendulum is a function of its length; the strength of an electromagnet is a function of the strength of the electrical current and of the number of windings of the wire; the properties of the chemical elements are functions of their atomic masses; the temperature at which water boils is a function of the atmospheric pressure, etc., etc.

EXERCISES XI.

1. If (i.) $f(x)=x^2$, form $f(x+h)$. *Ans.* $x^2+2xh+h^2$.

(ii.) $f(x)=\sin x$, form $f(x+h)$. *Ans.* $\sin(x+h)$.

(iii.) $f(x)=\log x^2$, form $f(x+h)$. *Ans.* $\log(\overline{x+h}^2)$.

(iv.) $f(x)=x^2+2x-5$, form $f(2)$. *Ans.* 3.

(v.) $\phi(x)=x^2+2$, form $\phi(a+b)$.

(vi.) $F(t)=\dfrac{t-1}{t^2+1}$, form $F(t+k)$.

2. If $F(y)=3y^3-2y^2+7y-9$, show that

(i.) $F(1)=-1$.

(ii.) $F(-1)=-21$.

(iii.) $F(2)=21$.

(iv.) $F(0)=-9$.

(v.) $F(a)=3a^3-2a^2+7a-9$.

(vi.) $F\left(\dfrac{b}{2}\right)=\dfrac{3b^3-4b^2+28b-72}{8}$.

(vii.) $F(y+h)=3y^3-2y^2+7y-9+(9y^2-4y+7)h+(9y-2)h^2+3h^3$.

3. If $\phi(z)=z^2-9z+20$, show that

(i.) $\phi(1)=\frac{3}{5}\phi(0)$.

(ii.) $\phi(4)=\phi(5)$.

(iii.) $\phi(-2)=7\phi(2)$.

(iv.) $\phi(z+2)=\phi(z)-\phi(6)-\phi(1)+4z$.

(v.) $\phi(z+k)=\phi(z)+(2z-9)k+k^2$.

4. If $f(t)=\dfrac{t+1}{t-1}$,

show that
$$\frac{1}{f(a)+f(b)}=\frac{ab-a-b+1}{2ab-2}.$$

5. If $\phi(x)=\log\dfrac{a+x}{a-x}$, show that
$$\phi(x)+\phi(y)=\phi\left\{\frac{a^2(x+y)}{a^2+xy}\right\}.$$

6. If $F(y)=y^{2n}+y^{2r}+1$, show that
$$F(a)=F(-a).$$

7. If $\phi(u)=u^{2n+1}+u^{2r+1}+u^3-5u$, show that
$$\phi(u)=-\phi(-u).$$

8. If $f(x)=\sin x$, show that
$$f(x)=-f(-x).$$

9. If $\psi(x) = \cos(3x)$, show that

$$\psi(x) = \psi(-x).$$

10. Assuming a curve as the graph of $y = f(x)$, what would be the graph of $y = f(-x)$?

ART. 9. **General rule for the formation of derivatives.** Let

$$(1) \qquad y = f(x)$$

be any function of x, and let the accompanying geometric curve (Fig. 37) be its graph. We take up the problem to find the tangent to the curve at any of its points.

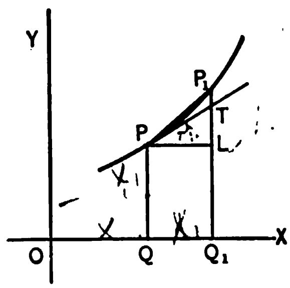

FIG. 37.

We use again the method employed, pp. 99–101. If we imagine a polygon having its vertices P, P_1, P_2, $\cdots$ lying on the curve, we can easily determine the angle α which the side $\overline{PP_1}$ of the polygon makes with the axis of abscissæ. We get

$$\tan\alpha = \frac{P_1L}{PL} = \frac{P_1Q_1 - PQ}{OQ_1 - OQ} = \frac{y_1 - y}{x_1 - x},$$

or, inasmuch as $y = f(x)$ and $y_1 = f(x_1)$,

$$(2) \qquad \tan\alpha = \frac{y_1 - y}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x}.$$

By putting $x_1 - x = h$, this quotient may be transformed so that it will contain only x and h, assuming the form

$$(3) \qquad \tan\alpha = \frac{f(x + h) - f(x)}{h}.$$

The angle τ, which the tangent at P makes with the axis of x, is the limit which the angle α approaches as h approaches zero.

We cannot actually determine the value of this limit unless the function in question is given, but can merely indicate it by the expression

$$\text{(4)} \qquad \lim_{h \doteq 0}\left[\frac{f(x+h)-f(x)}{h}\right].$$

Therefore the direction τ of the tangent at every point of the curve represented by equation (1) is given by the equation

$$\text{(5)} \qquad \tan\tau = \lim_{h \doteq 0}\left[\frac{f(x+h)-f(x)}{h}\right].$$

Denoting, as on p. 108, the difference of abscissæ by Δx, and the difference of ordinates by Δy; that is, putting

$$\text{(6)} \qquad x_1 - x = \Delta x, \quad y_1 - y = \Delta y,$$

we obtain

$$\text{(7)} \qquad \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x},$$

and this is the quotient whose limiting value for $h \doteq 0$, or for $\Delta x \doteq 0$, is to be determined. *The fraction represents the ratio between the increment of the function and the increment of the independent variable;* it is accordingly a measure of the greater or less rapidity with which the function increases or diminishes. The limit of this ratio is what, in previous instances, we have already called the **derivative**; or, expressed more exactly, it is the **derivative of y, or $f(x)$, with respect to x**, and

$$\frac{dy}{dx}, \ \frac{df(x)}{dx}, \ \frac{d}{dx}y, \ \frac{d}{dx}f(x)$$

are symbols each of which is often used to denote this limit.

The symbol $\frac{dy}{dx}$ is *not* a fraction, of which dy is the numerator and dx the denominator, but denotes the limit of the fraction $\frac{\Delta y}{\Delta x}$.

The symbol $\frac{d}{dx}$ placed before any function $f(x)$ denotes that the following *operation* is to be performed on that function:

First, the fraction $\frac{f(x+h)-f(x)}{h}$ is to be formed, and then the limit of this fraction as h approaches zero is to be taken.

We have accordingly the defining equation

$$\frac{df(x)}{dx} = \lim_{h \doteq 0}\left[\frac{f(x+h)-f(x)}{h}\right], \tag{8}$$

or we may also write it, putting y in place of $f(x)$ for brevity,

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \tag{9}$$

The result of the foregoing discussions concerning the tangent may now be stated thus: *For every curve whose equation is given in the form*

$$y = f(x),$$

the direction of the tangent at any of its points is determined by the equation

$$\tan \tau = \frac{dy}{dx} = \frac{df(x)}{dx}.$$

The definition of the derivative should be firmly fixed in mind, both as given in (8) and also as expressed in words: *The derivative of any function with respect to a variable is*

determined by means of that fraction whose numerator is the difference between the value of the function when the variable receives an increment and the value of the function as given, and whose denominator is the increment; the derivative is the limit which this fraction approaches as the increment approaches zero. Thus the derivative of x^3 is

$$\lim_{h \doteq 0} \frac{(x+h)^3 - x^3}{h};$$

that of $\sin x$ is

$$\lim_{k \doteq 0} \frac{\sin(x+k) - \sin x}{k}, \text{ etc.}$$

The letters used are, of course, immaterial.

This definition is fundamental for our whole subject; it gives us a *general scheme* or *rule* according to which to form the derivative of every function. One of the first problems that we shall solve is to determine the derivatives of the different simple functions. We *know* the derivative only in case we can *find* the limit indicated. If in any particular case no definite limit *exists*, the function in question has no derivative.

The process of finding the derivative of any given function is called **differentiation**.

EXERCISES XII

Write the defining expression for the derivatives of

$$x^7, \ \frac{1}{x}, \ (x-3)(x^2+5), \ \frac{x^2+2}{x-3}, \ \log y.$$

2. Find the derivative for the first three of the expressions in 1.

Of what functions, and with respect to what variables, are the following expressions derivatives? (Answer by inspection.)

3. $\lim_{h \doteq 0} \frac{\phi(x+h) - \phi(x)}{h}$.

4. $\lim_{k \doteq 0} \frac{\psi(u+k) - \psi(u)}{k}$.

5. $\lim_{h \doteq 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

6. $\lim_{l \doteq 0} \frac{F(y+l) - F(y)}{l}$.

7. $\lim\limits_{t \doteq 0} \dfrac{f(t+z)-f(z)}{t}$.

8. $\lim\limits_{h \doteq 0} \dfrac{\sin(x^2+h)-\sin x^2}{h}$.

9. $\lim\limits_{d \doteq 0} \dfrac{\log(x^2+2dx+d^2)-\log x^2}{d}$.

10. $\lim\limits_{y \doteq 0} \dfrac{\phi(a^3+a+y)-\phi(a^3+a)}{y}$.

11. $\lim\limits_{y \doteq 0} \dfrac{\phi\{(a+y)^3+a+y\}-\phi(a^3+a)}{y}$.

12. $\lim\limits_{c \doteq 0} \dfrac{\tan(x+c^2)-\tan x}{c^2}$.

13. $\lim\limits_{mr \doteq 0} \dfrac{f(x+mr)-f(x)}{mr}$.

14. $\lim\limits_{\lambda \doteq 0} \dfrac{\cos(u+\lambda^2+\lambda)-\cos u}{\lambda^2+\lambda}$.

15. $\lim\limits_{\Delta u \doteq 0} \dfrac{\phi(s+\Delta u)-\phi(s)}{\Delta u}$.

16. $\lim\limits_{\rho \doteq 0} \dfrac{f(x^2+\rho^2)-f(x^2)}{\rho^2}$.

17. Write the answers to **3** ⋯ **16** in the notation for derivatives explained above.

CHAPTER IV

DERIVATIVES OF THE SIMPLER FUNCTIONS

ART. 1. **The derivative of x^n.** The derivative of the expression x^n, n being a positive integer, is found as follows: The ratio

$$\frac{f(x+h)-f(x)}{h}$$

has in the present case the value

$$\frac{(x+h)^n - x^n}{h}$$

which by the binomial theorem is

$$(1)\quad \frac{\left(x^n + \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 + \cdots + \frac{n}{1}xh^{n-1} + h^n\right) - x^n}{h}.$$

This is the fraction whose limit is the derivative; x^n and $-x^n$ in the numerator cancel each other, and if h be taken out of the parenthesis,

$$(2)\quad \frac{(x+h)^n - x^n}{h}$$

$$= \frac{h\left\{\frac{n}{1}x^{n-1} + \frac{n(n-1)}{1.2}x^{n-2}h + \cdots + \frac{n}{1}xh^{n-2} + h^{n-1}\right\}}{h}$$

$$= \frac{n}{1}x^{n-1} + \frac{n(n-1)}{1.2}x^{n-2}h + \cdots + \frac{n}{1}xh^{n-2} + h^{n-1}.$$

If h be now made to approach the limit zero, the right member approaches the value nx^{n-1} as its limit; the derivative of x^n is therefore nx^{n-1}. We have thus the equation

$$(3)\qquad \frac{d(x^n)}{dx} = nx^{n-1}.$$

To illustrate, the derivative of x^2 is $2x$, that of x^3 is $3x^2$, etc. In particular, it follows that the derivative of x itself is 1, as is directly evident also, since when $n = 1$

$$\frac{x + h - x}{h} = 1.$$

ART. 2. **The derivative of $\sin x$ and of $\cos x$.** To obtain the derivative of $\sin x$, we first form the fraction whose limit is the derivative, *viz.*:

$$(1)\qquad \frac{\sin(x+h) - \sin x}{h} = \frac{2\sin\frac{x+h-x}{2}\cdot\cos\frac{x+h+x}{2}{}^{*}}{h}$$

$$= \frac{\sin\frac{h}{2}}{\frac{h}{2}}\cos\left(x + \frac{h}{2}\right).$$

But the limit of $\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}$, as h approaches zero, is unity,† and that of $\cos\left(x + \frac{h}{2}\right)$ is $\cos x$, and, accordingly, the right mem-

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* Formula 41, Appendix.

† This is usually proved in works on trigonometry. It may be proved as follows. In Fig. 38 it is seen without difficulty that

triangle $BOB' <$ sector $BAB'O <$ triangle TOT'.

ber of equation (1) has the limit $\cos x$, when h approaches the limit zero; hence the derivative of $\sin x$ is $\cos x$, or

(2) $$\frac{d \sin x}{dx} = \cos x.$$

In Fig. 39 is shown the curve whose equation is

$$y = \sin x.$$

Let the angle AOB contain $\frac{h}{2}$ radians (p. 74); then

$$BOB' = \tfrac{1}{2} BB' \cdot OC = BC \cdot OC = OB^2 \cdot \frac{BC}{OB} \cdot \frac{OC}{OB} = r^2 \sin\frac{h}{2} \cos\frac{h}{2};$$

$$\text{sector } BAB'O = \tfrac{1}{2} OB^2 \text{ arc } BB' = r^2 \frac{h}{2}.^*$$

$$TOT' = \tfrac{1}{2} TT' \cdot OA = AT \cdot OA = \overline{OA}^2 \cdot \frac{AT}{OA} = r^2 \tan\frac{h}{2}.$$

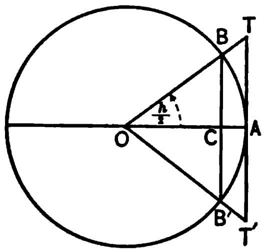

FIG. 38.

The inequality above therefore becomes

$$r^2 \sin\frac{h}{2} \cos\frac{h}{2} < r^2 \frac{h}{2} < r^2 \frac{\sin\frac{h}{2}}{\cos\frac{h}{2}},$$

or, after division by $r^2 \sin\frac{h}{2}$, into

$$\cos\frac{h}{2} < \frac{\frac{h}{2}}{\sin\frac{h}{2}} < \frac{1}{\cos\frac{h}{2}}.$$

* Formula 61, Appendix.

If it be borne in mind that (p. 117)

$$\tan \tau = \frac{dy}{dx} = \frac{d \sin x}{dx} = \cos x,$$

it is easily seen that to the values

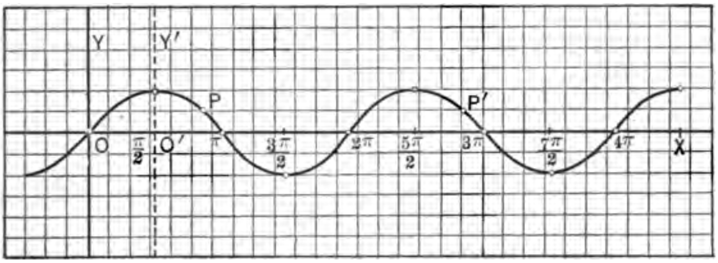

FIG. 39.

$$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \frac{5\pi}{2},\ 3\pi,\ \cdots,$$

there correspond the values

$$y = 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ \cdots,$$

and

$$\begin{aligned} \tan \tau &= \cos 0,\ \cos \frac{\pi}{2},\ \cos \pi,\ \cos \frac{3\pi}{2},\ \cos 2\pi,\ \cos \frac{5\pi}{2},\ \cos 3\pi,\ \cdots \\ &= 1,\quad 0,\quad -1,\quad 0,\quad 1,\quad 0,\quad -1. \end{aligned}$$

The middle fraction is the expression whose limit (or rather that of its reciprocal) is to be determined. The last relation shows that it always lies between

$$\cos \frac{h}{2} \text{ and } \frac{1}{\cos \frac{h}{2}},$$

the first being a proper, the second an improper fraction. If h approaches the limit zero, both

$$\cos \frac{h}{2} \text{ and } \frac{1}{\cos \frac{h}{2}}$$

approach the limit unity; and the ratio which is always intermediate must also approach unity; all three of the quantities thus approach the same limit.

That is, the curve cuts the x-axis at angles of 45° and 135° alternately. The equation

$$\sin(x + 2n\pi) = \sin x,$$

shows that the same value of y which corresponds to any value of x corresponds also to the value $x + 2n\pi$; we can therefore construct the entire curve by moving repeatedly the part extending from O to 2π, a distance equal to 2π either to the right or left.

The curve is a simple *periodic curve*, and is called the **sine-curve.**

The derivative of $\cos x$ is obtained in a manner similar to the above. We have*

$$(3)\quad \frac{\cos(x+h) - \cos x}{h} = -\frac{2\sin\frac{x+h-x}{2}\cdot\sin\frac{x+h+x}{2}}{h}$$

$$= -\sin\left(x + \frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}},$$

and the limit of this expression as h approaches zero is $-\sin x$; the derivative of $\cos x$ is $-\sin x$; that is,

$$(4)\qquad \frac{d\cos x}{dx} = -\sin x.$$

Art. 3. **Geometric interpretation of the sign of the derivative.** It is of interest to determine the significance of the negative sign in the last equation. We know that the derivative is the limiting value of the ratio of $\Delta\cos x$ to Δx, where $\Delta\cos x$ indicates the increment that $\cos x$ receives

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* Formula 43, Appendix.

when x increases by Δx. This increment is negative; *i.e.* $\cos x$ at first decreases when the arc x increases;* and as a matter of fact $\cos 0 = 1$ and $\cos \frac{\pi}{2} = 0$.

The above statement holds for every function whose derivative is negative; it can be enunciated in the form of the following theorem:

If a function increases continually for a sequence of increasing values of x, its derivative for these values of x is positive; but if, on the other hand, it decreases continually, its derivative is negative.

This fact may be illustrated in the following manner: Let

(1) $$y = f(x)$$

be a function whose graphic representation is the accompanying curve (Fig. 40). We have for this curve

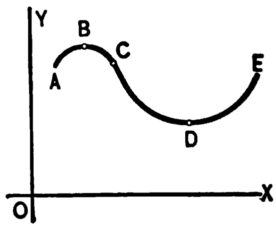

FIG. 40.

(2) $$\tan \tau = \frac{dy}{dx} = \frac{df(x)}{dx}.$$

If B is the highest and D the lowest point of the curve, the ordinate (*i.e.* the function), increases from A to B and from D to E, and we easily see that along these portions of the curve the angle τ is acute and $\tan \tau$ is positive; on the other hand, along the portion of the curve BCD, the ordinate or the function continuously decreases so that in this

* This is true as long as the arc x lies between 0 and π. If $x > \pi$, $\sin x$ becomes negative, and therefore $\Delta \cos x$, positive again, etc. Thus equation (4) agrees completely with the fact that in the first quadrant the cosine diminishes continually from unity to zero, and in the second quadrant from zero to minus one, while in the third quadrant it *increases* from minus one to zero, and in the fourth quadrant from zero to plus one.

case τ is an obtuse angle, and $\tan\tau$ is accordingly negative, and at the points B and D the tangent is parallel to the x-axis and $\tan\tau$ is zero.

Exercise. Construct the graph for $y=\cos x$ and discuss it as $y=\sin x$ was discussed above. Show that the curve of Fig. 39 will represent $\cos x$ if the origin be shifted $\frac{\pi}{2}$ radians along the x-axis, $O'Y'$ in the figure being taken as the y-axis.

ART. 4. **Derivatives of sums and differences.** If $f(x)$ and $\phi(x)$ are two functions whose derivatives are known, the derivative of their sum is found in the following manner:

We form

$$(1)\qquad \frac{[f(x+h)+\phi(x+h)]-[f(x)+\phi(x)]}{h}$$
$$=\frac{f(x+h)-f(x)}{h}+\frac{\phi(x+h)-\phi(x)}{h};$$

the limit of (1), when h approaches the limit zero, is, by definition, the derivative sought; and the limits of the fractions in the right member are the derivatives of $f(x)$ and $\phi(x)$, respectively; we have then

$$(2)\qquad \frac{d[f(x)+\phi(x)]}{dx}=\frac{df(x)}{dx}+\frac{d\phi(x)}{dx}.$$

In words, *the derivative of the sum of two functions is equal to the sum of their derivatives.*

It is apparent that this holds good for any number of functions, or

$$(3)\qquad \frac{d}{dx}[f(x)+\phi(x)+\psi(x)+\cdots]$$
$$=\frac{df(x)}{dx}+\frac{d\phi(x)}{dx}+\frac{d\psi(x)}{dx}+\cdots.$$

The derivative of the difference of two functions is obtained in a similar manner. Here we form

$$(4)\qquad \frac{[f(x+h)-\phi(x+h)]-[f(x)-\phi(x)]}{h}$$

$$=\frac{f(x+h)-f(x)}{h}-\frac{\phi(x+h)-\phi(x)}{h},$$

and allowing h to approach the limit zero, we have

$$\frac{d[f(x)-\phi(x)]}{dx}=\frac{df(x)}{dx}-\frac{d\phi(x)}{dx}.$$

In words, *the derivative of the difference of two functions is equal to the difference of their derivatives.*

For brevity, single letters, u, v, w, $\cdots$ are often used to denote functions of x instead of $f(x)$, $\phi(x)$, $\psi(x)$, $\cdots$, and with this notation the above results may be stated in the more compact form,

$$\frac{d(u+v+w+\cdots)}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\cdots,$$

$$\frac{d(u-v)}{dx}=\frac{du}{dx}-\frac{dv}{dx}.$$

EXAMPLES

1. $\dfrac{d(x+\sin x)}{dx}=\dfrac{d}{dx}x+\dfrac{d\sin x}{dx}=1+\cos x.$

2. $\dfrac{d(x^2-\cos x)}{dx}=\dfrac{d(x^2)}{dx}-\dfrac{d(\cos x)}{dx}=2x+\sin x.$

3. $\dfrac{d(x^3+x^2-x)}{dx}=\dfrac{d(x^3)}{dx}+\dfrac{d(x^2)}{dx}-\dfrac{dx}{dx}=3x^2+2x-1.$

ART. 5. **The derivative of $cf(x)$, c being a constant.** We form

$$(1) \qquad \frac{cf(x+h)-cf(x)}{h}=c\frac{f(x+h)-f(x)}{h},$$

and taking the limits as h approaches zero, we obtain

$$(2) \qquad \frac{d[cf(x)]}{dx}=c\frac{df(x)}{dx};$$

the constant is thus seen to become a factor of the derivative. For example, the derivatives of

$$ax^n, \quad b\sin x, \quad c\cos x,$$

are $\qquad nax^{n-1}, \quad b\cos x, \quad -c\sin x$, respectively.

ART. 6. **The derivative of a constant.** What is the derivative of a function that is known always to have the same value? Let

$$y=f(x)$$

be a function such that for all the values of the variable x, y has the same value; the numerator of the fraction

$$\frac{f(x+h)-f(x)}{h}$$

is then equal to zero for every value of x or h; the fraction is therefore always zero, and hence its limit must also be zero, and we have, if c represent a constant,

$$(1) \qquad \frac{dc}{dx}=0.$$

In words, *the derivative of a constant is zero.*

This conclusion can be illustrated geometrically. Inasmuch as the function y may be represented by an equation of the form

$$y=b,$$

the curve corresponding to this equation must be a straight line parallel to the axis of abscissæ, and consequently the angle which it makes with the axis of x, as well as the tangent of this angle, must be equal to zero; that is,

$$\frac{dy}{dx} = 0.$$

EXAMPLES

1. $\frac{d(5x^2 + 3x - 1)}{dx} = \frac{d(5x^2)}{dx} + \frac{d(3x)}{dx} - \frac{d(1)}{dx} = 5\frac{d(x^2)}{dx} + 3\frac{dx}{dx} = 10x + 3.$

2. $\frac{d(7x^3 - 6x^2 + 4)}{dx} = \frac{d(7x^3)}{dx} - \frac{d(6x^2)}{dx} = 21x^2 - 12x.$

3. $\frac{d(ax + b\sin x + c\cos x)}{dx} = a + b\cos x - c\sin x.$

4. If $y = ax^n + bx^{n-1} + cx^{n-2} + \cdots + px^2 + qx + r$, wherein $a, b, c, \cdots p, q, r$ are constant quantities, then

$$\frac{dy}{dx} = nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \cdots + 2px + q.$$

EXERCISES XIII

Find the derivative of each of the following expressions:

(In this, and the other sets of exercises, the first portion of the exercises can usually be solved without the use of pencil and paper. It is recommended that this be done. The number of exercises which can be solved thus will vary with different persons. Recourse should always be had to written work whenever it becomes confusing to hold the computations in the mind. On review it should be possible to solve a large number without pencil and paper.)

1. x^4.
2. x^7.
3. $3x^6$.
4. $2x^5$.
5. $\frac{x^{10}}{10}$.
6. $-x^4$.
7. $-\frac{x^6}{3}$.
8. $\frac{12x}{17}$.
9. $(a-b)x$.
10. $\frac{4x^3}{3}$.
11. $-ax^2$.
12. $3bx^6$.
13. $\frac{4x^c}{c}$.
14. $6x^6$.
15. $4x^3$.
16. nx^n.
17. $2x^{2n}$.
18. $\frac{x^{2n}}{n}$.
19. x^{a+b}.
20. $(c+d)x^{c-d}$.
21. $\frac{2x^9}{3}$.
22. $x - 5$.

23. $x^2 + 3x$.

24. $2x^2 - 5x + 4$.

25. $5x^2 + 10x - 3$.

26. $x^3 - 1$.

27. $2x^3 + 3x^2$.

28. $x^5 - 5x$.

29. $\dfrac{x^{24} + x^{12}}{12}$.

30. $x^a + ax$.

31. $x^{a+b} - x^{a-b}$.

32. $\dfrac{x^c + x^d}{cd}$.

33. $\sin x - 2$.

34. $\sin x + \cos x$.

35. $2 \sin x - 5 \cos x$.

36. $x + \cos x$.

37. $4 + \cos x$.

38. $3x - 5 \sin x$.

39. $12x^2 - 12 \sin x + \cos x$.

40. $ax^3 + bx^2 + cx + d$.

41. $ax^4 + bx^3 + cx^2 + dx + e$.

42. $ax^4 - 4bx^3 + 6cx^2 - 4dx + e$.

43. $m \cos x - r \sin x$.

44. $(a - b) \cos x + (b - a) \sin x$.

ART. 7. **The derivative of a product.** To obtain the derivative of the product of two functions $f(x)$ and $\phi(x)$, we alter the quotient,

$$\text{(1)} \qquad \frac{f(x+h) \cdot \phi(x+h) - f(x) \cdot \phi(x)}{h},$$

whose limit when $h \doteq 0$ is, by definition, the derivative which we seek, by adding and subtracting in the numerator the quantity $f(x) \cdot \phi(x + h)$, obtaining thus

$$\frac{f(x+h) \cdot \phi(x+h) - f(x) \cdot \phi(x)}{h}$$

$$= \frac{f(x+h)\phi(x+h) - f(x)\phi(x+h) + f(x)\phi(x+h) - f(x)\phi(x)}{h}$$

$$= \phi(x+h)\frac{[f(x+h) - f(x)]}{h} + f(x)\frac{[\phi(x+h) - \phi(x)]}{h}.$$

If we now allow h to approach the limit zero, we have

$$\text{(2)} \qquad \frac{d[f(x) \cdot \phi(x)]}{dx} = \phi(x)\frac{df(x)}{dx} + f(x)\frac{d\phi(x)}{dx};$$

or, on introducing a more compact notation,

$$\text{(3)} \qquad \frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

In words, *the derivative of a product of two factors, is the first factor into the derivative of the second, plus the second factor into the derivative of the first.*

If the product whose derivative is to be determined contains more than two factors, it may be divided up in some way or other into two factors before differentiation.

EXAMPLES

1. $\dfrac{d(x\sin x)}{dx} = \sin x\dfrac{dx}{dx} + x\dfrac{d\sin x}{dx} = \sin x + x\cos x.$

2. $\dfrac{d(\sin x\cos x)}{dx} = \cos x\dfrac{d\sin x}{dx} + \sin x\dfrac{d\cos x}{dx} = \cos^2 x - \sin^2 x.$

3. $\dfrac{d(ax^2\cos x)}{dx} = a(2\,x\cos x - x^2\sin x).$

4. $\dfrac{d(x^3\sin x + a\cos x)}{dx} = 3\,x^2\sin x + x^3\cos x - a\sin x.$

5. Given the function $x^2\sin x\cos x$, we find, on taking x^2 as one factor and $\sin x\cos x$ as the other, that

$$\begin{aligned}\frac{d(x^2\sin x\cos x)}{dx} &= \sin x\cos x\frac{d(x^2)}{dx} + x^2\frac{d(\sin x\cos x)}{dx}\\ &= 2\,x\sin x\cos x + x^2(\cos^2 x - \sin^2 x),\text{ by } \mathbf{2}.\end{aligned}$$

To deduce corresponding formulæ for the case of three factors, we have

$$\frac{d(uvw)}{dx} = vw\frac{du}{dx} + u\frac{d(vw)}{dx} = vw\frac{du}{dx} + u\left(w\frac{dv}{dx} + v\frac{dw}{dx}\right);$$

that is,
$$\frac{d(uvw)}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}.$$

We observe that the derivative of a product of three factors is the sum of the derivatives of each factor multiplied by the other two factors. It is easily seen similarly that the derivative of the product of k factors is the sum of k terms.

each of which consists of all the factors save one multiplied by the derivative of that one factor, the derivative of any factor occurring in one and only one term. The formal proof may be supplied by the student.

EXERCISES XIV

Find the derivatives of the following expressions:

1. $y = (x + 2)(x - 3)$.*
2. $y = \sin x \cos x$.
3. $y = \sin^2 x \,(= \sin x \cdot \sin x)$.
4. $y = x^2 \cos x$.
5. $y = (4x^2 + 1)(3x^3 - 5)$.
6. $y = \cos^2 x$.
7. $y = \cos^3 x \,(= \cos x \cdot \cos^2 x)$.
8. $y = \cos^4 x$.
9. $y = \cos^2 x \sin^2 x$.
10. $y = x^3 \cos^2 x$.
11. $y = \cos 2x \,(= \cos^2 x - \sin^2 x)$.
12. $y = (x^2 + 1)(x^3 + 2)(x^4 + 3)$.

ART. 8. **The derivative of a quotient.** We now proceed to deduce the derivative of the quotient of two functions. At once denoting the two functions by u and v, and putting

$$y = \frac{u}{v}, \tag{1}$$

we get

$$u = yv,$$

and on forming the derivative of both of its members, we find

$$\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}, \tag{2}$$

from which the required value of $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{1}{v}\left(\frac{du}{dx} - y\frac{dv}{dx}\right). \tag{3}$$

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* Though not necessary, it is often convenient to use a single letter to denote the expression to be differentiated. Of course, y is simply another name for the expression on the right in each case.

On substituting the value of y in the right-hand member, we have

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

or

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{4}$$

In words: *the derivative of the quotient of two functions is the denominator into the derivative of the numerator minus the numerator into the derivative of the denominator all divided by the square of the denominator.* The key word *denominator* helps to make the above easy to remember.

I. To determine the derivative of $\tan x$ and $\cot x$. Since

$$\tan x = \frac{\sin x}{\cos x}, \tag{5}$$

we have in the case in hand, $u = \sin x$, whence $\dfrac{du}{dx} = \cos x$, and $v = \cos x$, whence $\dfrac{dv}{dx} = -\sin x$; then

$$\frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},\text{*}$$

or

$$\frac{d \tan x}{dx} = \frac{1}{\cos^2 x}; \tag{6}$$ $= \sec^2 X$

i.e. *the derivative of* $\tan x$ *is* $\dfrac{1}{\cos^2 x}$.

Since

$$\cot x = \frac{\cos x}{\sin x}, \tag{7}$$

* Formula 28, Appendix.

in this case $u = \cos x$, and $v = \sin x$, and accordingly,

$$\frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x},$$

or (8) $$\frac{d \cot x}{dx} = -\frac{1}{\sin^2 x}; \quad -\csc^2 x,$$

i.e. *the derivative of* $\cot x$ *is* $-\dfrac{1}{\sin^2 x}$.

II. Further, let $y = \dfrac{a}{x}$, where a is a constant; here $u = a$, and $v = x$; and $\dfrac{du}{dx} = 0$, $\dfrac{dv}{dx} = 1$;
whence

(9) $$\frac{dy}{dx} = -\frac{a}{x^2}.$$

III. Likewise,

(10) $$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.$$

Similarly,

(11) $$\frac{d}{dx}\operatorname{cosec} x = -\cot x \operatorname{cosec} x.$$

IV. $\dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)$

$$= \frac{(a-x)\dfrac{d(a+x)}{dx} - (a+x)\dfrac{d(a-x)}{dx}}{(a-x)^2} = \frac{2a}{(a-x)^2}.$$

V. $$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\sin x\dfrac{dx^2}{dx} - x^2\dfrac{d\sin x}{dx}}{\sin^2 x} = \frac{2x\sin x - x^2\cos x}{\sin^2 x}$$

VI. According to Boyle's Law (p. 3), we have for the volume v corresponding to the pressure p, the equation

$$vp = v_0 p_0,$$

where p_0 and v_0 stand for the initial pressure and volume, respectively. By writing this equation in the form

$$v = \frac{v_0 p_0}{p},$$

we obtain in accordance with II,

$$\frac{dv}{dp} = -\frac{v_0 p_0}{p^2}.$$

The derivative is negative; it is also (p. 108) the limiting value of the ratio of Δv to Δp, which represent the increments of volume and of pressure; the negative sign indicates that as the pressure increases the volume decreases (p. 125). The relation between the decrease in volume to the increase in pressure is according to our equation inversely proportional to p^2; for

$$p = 2, 3, 4, \cdots$$

this relation is proportional to

$$\tfrac{1}{4}, \tfrac{1}{9}, \tfrac{1}{16}, \cdots.$$

If the pressure be greatly increased, the decrease of volume soon becomes very slight, a conclusion in perfect accordance with the experimental observation of gases.

EXERCISES XV

Find the derivatives of the following expressions:

1. $\dfrac{x+1}{x+2}$.
2. $\dfrac{1}{x}$.
3. $\dfrac{3}{x^4}$.
4. $-\dfrac{1}{5x^5}$.
5. $\dfrac{x^2+3}{x^3-1}$.
6. $\dfrac{x \sin x}{2x-3}$.
7. $\dfrac{1}{a^2 - ax + x^2}$.
8. $\dfrac{\sin x}{a + b\cos x}$.
9. $\dfrac{x^2+4x-2}{x^2-4x+2}$.
10. $\dfrac{\sin x + \tan x}{\cos x}$.
11. $x \tan x$.
12. $-\dfrac{1}{x^n}$.
13. $\dfrac{1}{cx^c}$.
14. $\dfrac{x^{10}}{1+x^{10}}$.
15. $\cos x \cot x$.
16. $\dfrac{\sec x}{1+\cos x}$.
17. $\dfrac{x^5 - x^4 + 7}{5x^4 - 4x^3}$.

Art. 9. **Logarithmic functions.** Our conception of logarithms consists in regarding all numbers as powers of a fundamental number, the **base**; the exponent, which indicates what power of the base equals the number in question, is called the **logarithm** of the latter.

The tables of logarithms in general use take the number 10 as base, because of the advantages thus obtained in numerical calculations with logarithms; we shall see later that in theoretic mathematics there is an advantage in using a system of logarithms with another base. In what follows immediately we leave the base of the system of logarithms undetermined.

In order to obtain the logarithmic derivative, we have to find the limit, when h approaches the limit zero, of the quotient

$$(1) \qquad \frac{\log(x+h)-\log x}{h}=\frac{1}{h}\log\frac{x+h}{x}{}^{*}=\frac{1}{h}\log\left(1+\frac{h}{x}\right),$$

$$(2) \qquad =\log\left\{\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right\}.\dagger$$

The right member of this equation is not in a form which permits its limit to be discerned immediately, but requires a somewhat long discussion. We put

$$(3) \qquad \frac{h}{x}=\frac{1}{\delta}, \text{ that is, } \frac{1}{h}=\frac{\delta}{x},$$

and then have

$$\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\left(1+\frac{1}{\delta}\right)^{\frac{\delta}{x}}=\left\{\left(1+\frac{1}{\delta}\right)^{\delta}\right\}^{\frac{1}{x}};\ddagger$$

---

* Formula 6, Appendix. † Formula 8, Appendix.
‡ Formula 2, Appendix.

Substituting in (2), we obtain finally

$$(4) \qquad \frac{\log(x+h)-\log x}{h}=\frac{1}{x}\log\left(1+\frac{1}{\delta}\right)^{\delta}.$$

We have next to determine the limit of

$$\log\left(1+\frac{1}{\delta}\right)^{\delta}, \text{ or of } \left(1+\frac{1}{\delta}\right)^{\delta},$$

when h approaches the limit zero. Equation (3) shows that when h is approaching the limit zero, δ, on the other hand, is continually increasing; we have then to find the limit of the above expression when δ increases without bound. To simplify matters, we assume at first that δ is always a positive integer.* We have then according to the Binomial Theorem,†

$$(5)\left(1+\frac{1}{\delta}\right)^{\delta}=1+\frac{\delta}{1}\cdot\frac{1}{\delta}+\frac{\delta(\delta-1)}{1.2}\left(\frac{1}{\delta}\right)^2+\frac{\delta(\delta-1)(\delta-2)}{1.2.3}\left(\frac{1}{\delta}\right)^3+\cdots$$

$$=1+\frac{1}{1}+\frac{1-\frac{1}{\delta}}{1\cdot 2}+\frac{\left(1-\frac{1}{\delta}\right)\left(1-\frac{2}{\delta}\right)}{1\cdot 2\cdot 3}+\cdots.$$

Since δ is a positive integer, the right member contains $\delta+1$ terms.

We now seek to find the limit of the right member when h approaches the limit zero. In this case $\frac{1}{\delta}$ also approaches the limit zero, and by inspection of the right member of (5),

* It can be proved that all the following conclusions are true, even without such an assumption, which we make only to render our treatment simpler.

† Formula 3, Appendix.

we see that under these circumstances it approaches the limit

$$(6) \qquad 1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.$$

The limit of the left member of (5) is designated by e; that is, we put

$$(7) \qquad e=1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots;$$

the number e thus defined plays just as important a rôle in mathematics as does the number π.

Like π, it can be calculated only approximately. Its value to the tenth decimal place is

$$e=2.7182818284.$$

The calculation as based upon the above equation is very simple. We find

$$\begin{aligned} 1+\frac{1}{1}&=2\\ \frac{1}{2!}&=0.5\\ \frac{1}{3!}=\frac{1}{2!}:3&=0.16667\\ \frac{1}{4!}=\frac{1}{3!}:4&=0.04167\\ \frac{1}{5!}=\frac{1}{4!}:5&=0.00833\\ \frac{1}{6!}=\frac{1}{5!}:6&=0.00139\\ \hline 1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}&=2.71806\cdots, \end{aligned}$$

and thus obtain the first three decimal places exactly. The series is accordingly very well adapted to the approximate calculation of the actual value of e.

. We have then

$$\lim_{\frac{1}{\delta} \doteq 0}\left(1 + \frac{1}{\delta}\right)^{\delta} = e,$$

and substituting this result in (4), we find

$$(8) \qquad \frac{d \log x}{dx} = \frac{1}{x} \log e.$$

Thus far we have made no decision as to what base of our system of logarithms we are to adopt. If we take 10 as the base, the corresponding value of $\log e$ is

$$\log_{10} e = 0.43429 \cdots.$$

The derivative assumes the simplest form, however, when

$$\log e = 1;$$

that is, when the system of logarithms has e as its base.

The fundamental notion of logarithms as applied in abbreviating numerical calculations was first formulated and published by Baron Napier of Merchiston (in Scotland) in his *Mirifici Logarithmorum Canonis Descriptio*, 1614. Though Napier himself did not devise logarithms to the base e, and indeed did not build his theory upon the notion of any base, yet he furnished the impetus and the fundamental idea which speedily led to the setting up of systems of logarithms to the base e as well as to the base 10, as we now have them, and accordingly, in honor of this great invention, logarithms to the base e are often called **Napierian logarithms.** They are also called **natural logarithms,** because the theory of many problems may be discussed more simply when these logarithms are employed.

Logarithms to the base 10 are called **Briggean logarithms,** in honor of Henry Briggs, a contemporary of Napier, who proposed this base, and also **common logarithms,** because they are used almost exclusively in practical computations.

In what follows we shall usually employ natural logarithms. They are denoted by $\log \text{nat}\, x$, or, more briefly, by $\log x$; we shall use the briefer symbol, and shall always

understand by $\log x$ the natural logarithm of x, while logarithms to any other base, as a, will be denoted by $\log_a$.* Referring to equation (8), we have then the following formulæ:

(9)
$$\frac{d \log x}{dx} = \frac{1}{x},$$

(10)
$$\frac{d \log_a x}{dx} = \frac{1}{x} \log_a e.$$

ART. **10. Relations between logarithms with different bases.** If

$$a^m = x \text{ and } b^r = x,$$

we have, in accordance with the definition of logarithms,

$$m = \log_a x, \quad r = \log_b x.$$

Further, we have from

$$a^m = b^r,$$

by taking logarithms to the base a on each side,

$$m = r \log_a b.$$

By substituting in this equation the values of m and r, we find

$$\log_a x = \log_b x \log_a b,$$

or

(1)
$$\log_b x = \frac{\log_a x}{\log_a b}.$$

This equation furnishes us with a means of calculating the logarithm of any given number for the base b, when we know its logarithm for the base a.

---

* The notation $\log x$ is that usually employed by English and American writers, while $\ln x$ is used by Continental writers.

In particular, if $a = 10$ and $b = e$, so that $\log_a x$ represents the common logarithm and $\log_b x$ the natural logarithm, formula (1) passes into

$$(2) \qquad \log_{10} x = \frac{\log_e x}{\log_e 10}.$$

Thus, when the natural logarithm of x is known, we obtain the Briggean logarithm by multiplying the former logarithm by a constant factor $\frac{1}{\log_e 10}$, which may be computed once for all. It is known as the **modulus** of the Briggean logarithms, is denoted by M, and has the value $M = 0.43429 \cdots$.

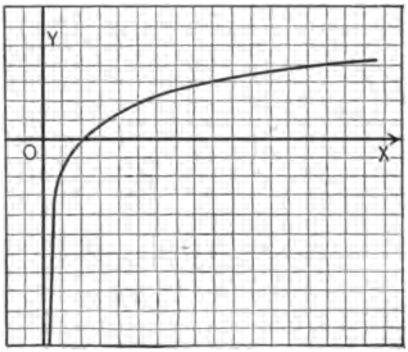

FIG. 41.

In conclusion, we give the graphic representation of the natural logarithm (Fig. 41);* that is, of the equation

$$(3) \qquad y = \log_e x.$$

We obtain the following table of corresponding values of x and y, as well as of $\tan \tau$,

$$x = 0, \quad \frac{1}{e^2}, \quad \frac{1}{e}, \quad 1, \quad e, \quad e^2, \quad \infty,$$

$$y = -\infty, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad \infty,$$

$$\tan \tau = \infty, \quad e^2, \quad e, \quad 1, \quad \frac{1}{e}, \quad \frac{1}{e^2}, \quad 0.$$

---

* The unit of length consists of two of the spaces into which the x-axis is divided in the figure.

As there are no logarithms of negative numbers, no points of the curve can correspond to negative values of x, and the accompanying figure is the graphic representation for the logarithm of x. It shows that as x increases from 1 to ∞ the logarithm also increases to ∞, but very slowly, while, on the other hand, as x decreases from 1 to 0, it decreases very rapidly from 0 to $-\infty$. Moreover, the angle which the tangent of the curve makes with the axis of abscissæ, is 45° at the point where $x = 1$ and $y = 0$; as x diminishes from this point, the angle increases approaching the limit 90°, as x approaches zero; as x increases from this point, the angle approaches zero, as x grows beyond all bounds. The curve intersects the axis of abscissæ at an angle of 45°, and it has the y-axis as an asymptote.

ART. 11. **Connection between $\frac{dy}{dx}$ and $\frac{dx}{dy}$.** Suppose we have

(1) $$y = f(x),$$

and from this, expressing x in terms of y,

(2) $$x = \phi(y).$$

(3) Then $$\frac{dy}{dx} = \lim_{h_1 \doteq 0} \frac{f(x + h_1) - f(x)}{h_1},$$

(4) $$\frac{dx}{dy} = \lim_{h_2 \doteq 0} \frac{\phi(y + h_2) - \phi(y)}{h_2}.$$

Here h_1 and h_2 may be quantities of any form provided they can be made to approach zero. Accordingly, we choose

$$h_2 = f(x + h_1) - f(x),$$

or, by (1),

(5) $$h_2 = f(x + h_1) - y.$$

We see that when h_1 approaches zero, h_2 also approaches zero,* and therefore, by (4) and (5),

(6) $$\frac{dx}{dy} = \lim_{h_1 \doteq 0} \frac{\phi[f(x+h_1)] - \phi(y)}{f(x+h_1) - f(x)}.$$

Let y_1 be the result of substituting $x + h_1$ for x in equation (1), so that

(7) $$y_1 = f(x + h_1).$$

But since equation (2) is equivalent to (1), equation (2) must be satisfied by the same values y_1 or $x + h_1$ which satisfy (1), and we have

(8) $$x + h_1 = \phi[f(x + h_1)].$$

Now, by means of equations (8) and (2), equation (6) becomes

(9) $$\frac{dx}{dy} = \lim_{h_1 \doteq 0} \frac{x + h_1 - x}{f(x+h_1) - f(x)}$$
$$= \lim_{h_1 \doteq 0} \frac{h_1}{f(x+h_1) - f(x)}.$$

Comparing (9) with (3), we have, finally,

(10) $$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

(11) or $$\frac{dx}{dy} \cdot \frac{dy}{dx} = 1.$$

ART. **12. The exponential function.** From the equation

(1) $$x = \log_e y$$

there can be at once derived the equation

(2) $$y = e^x.$$

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* The hypothesis is here, as always, tacitly made that $f(x)$ is a *continuous* function of x; see pp. 160–165.

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We have thus obtained a new function e^x, which may be regarded as the inverse of the logarithmic function. This function, *i.e.*, the xth power of e, in which x occurs as an exponent, is called the **exponential function.**

We obtain its derivative by differentiating equation (1) with respect to y, with the result

$$\frac{dx}{dy}=\frac{1}{y};$$

or, in accordance with p. 143,

$$\frac{dy}{dx}=y.$$

Substituting for y its value from equation (2), we have

$$\frac{de^x}{dx}=e^x. \tag{3}$$

In words, *the derivative of the exponential function is identical with the function itself.*

Art. **13. Illustrative discussion of the exponential function.** The curve of Fig. 41 (p. 141) may be regarded as the graphic representation of the exponential function; for from equation (1), p. 143, we have

$$x=e^y$$

and if in this, we interchange the letters x and y,

$$y=e^x.$$

Hence, by supposing Fig. 41 to be turned so that the positive portions of the axes are interchanged, we obtain the graphic representation of the exponential function. While x passes from 0 to ∞, e^x increases with great rapidity from 1 to ∞; as x passes from 0 to $-\infty$, the exponential function decreases

very slowly from 1 to 0. For every positive or negative value of x the corresponding value of y is positive in accordance with the fact that $e^{-x} = 1 : e^x$. The same is true of $\tan \tau$, inasmuch as

$$(1) \qquad \tan \tau = \frac{dy}{dx} = e^x.$$

The significance of the exponential function in the phenomena of nature is illustrated by the following discussion. If a capital of c dollars be invested at $p\,\%$ interest for one year, the capital, together with the interest, will be equal to

$$(2) \qquad c_1 = c + c\frac{p}{100} = c\left(1 + \frac{p}{100}\right).$$

If the capital c_1 draw interest for another year at the same rate, the sum of the capital and interest at the end of the second year will amount to

$$(3) \qquad c_2 = c_1 + c_1 \cdot \frac{p}{100} = c_1\left(1 + \frac{p}{100}\right) = c\left(1 + \frac{p}{100}\right)^2.$$

If this be continued for n years, it is readily seen that the capital will be

$$(4) \qquad c_n = c\left(1 + \frac{p}{100}\right)^n.$$

If we now suppose that the interest is added to the capital every month, and thus the money bearing interest is increased every month, the sum of the capital and interest at the end of the first month will be

$$c_1 = c + c \cdot \frac{p}{100.12} = c\left(1 + \frac{p}{12.100}\right),$$

and at the end of the second month it will be

$$c_2 = c_1\left(1 + \frac{p}{12.100}\right) = c\left(1 + \frac{p}{12.100}\right)^2,$$

and so on; in a year the value of the capital will be

(5) $$c_{12} = c\left(1 + \frac{p}{12.100}\right)^{12}.$$

It is easy to see how the formula will become altered if the interest be added to the capital every day or every hour. As the time is made shorter and shorter, we approach more nearly to what actually occurs in nature. When in any process of nature, an active force increases continuously from its own action, the force added at each instant immediately has all the effects and powers of the original force, and exerts them conjointly with it. In order to extend the above formula to the case when the interest is added continually to the working or interest-bearing capital, we must substitute for the number 12, a number n, that increases indefinitely; if besides, we write q instead of $\frac{p}{100}$, and let C denote the amount of capital and interest at the close of the year, we have

$$C = c\left[\lim_{n \doteq \infty}\left(1 + \frac{q}{n}\right)^n\right].$$

If we now put

$$\frac{q}{n} = \frac{1}{\delta}, \quad \text{or} \quad n = \delta q,$$

we have

$$\left(1 + \frac{q}{n}\right)^n = \left(1 + \frac{1}{\delta}\right)^{\delta q} = \left[\left(1 + \frac{1}{\delta}\right)^{\delta}\right]^q, *$$

and we find finally, by taking the limit of this expression as n (and consequently δ) increases without bound, that

(6) $$C = ce^q.$$

The exponential function e^q thus determines the amount of an active or interest-bearing capital after a year's time

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* Formula 2, Appendix.

(or any other given unit of time), it being assumed that the increase at any instant is proportional to the active capital, the number q being the proportion of increase in the unit of time, provided the increase is not added to the active capital.*

ART. **14. Inverse trigonometric functions.** If

(1) $$x = \sin y,$$

y may also be regarded as a function of x; y is an angle whose sine is x. The notation is†

(2) $$y = \text{arc}(\sin x) = \text{arc} \sin x,$$

which indicates that y is an angle whose sine is x.

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* This result is due to James Bernoulli (1654–1705), the first of a remarkable Swiss family group, who *would* be mathematicians despite all obstacles. In accordance with the wishes of his father, James studied Divinity; at the same time, following his own inclinations, he quietly devoted himself to mathematics and astronomy. Refusing a pastorate which was offered him after the completion of his studies and travels, he settled in Basel as professor of mathematics and physics. Among his pupils was his younger brother, John Bernoulli (1667–1748), for whom also mathematics had irresistible attractions, overthrowing his father's plan, which destined him for a commercial career. Upon the death of James Bernoulli, John succeeded to his professorship, and held it until his own death, over forty years later. A nephew, and pupil of both the brothers in turn, Nicholas Bernoulli (1687–1759), professor of mathematics at Padua, and later of logic at Basel, and two sons of John Bernoulli, — Nicholas Bernoulli, the second (1695–1726), and Daniel Bernoulli (1700–1782), both for a time members of the newly founded academy at St. Petersburg, — complete this family group of five men, who for a whole century were prominent figures in the mathematical world and leaders in its activities. We may mention also a third son of John Bernoulli, *viz.* John Bernoulli, the second (1710–1790), who devoted himself to physics; also a son of the latter, John Bernoulli, the third (1744–1807), who made some contributions to the history of mathematics.

† Many English and American writers use the notation $\sin^{-1} x$, etc., instead of arc sin x, etc.

In a similar manner

(3) $$x = \cos y,$$

where y is an angle whose cosine is x, yields

(4) $$y = \text{arc}(\cos x) = \text{arc} \cos x,$$

and likewise,

(5) $x = \tan y$, $x = \cot y$, $x = \sec y$, and $x = \text{cosec}\, y$

yield the inverse functions,

(6) $$y = \text{arc} \tan x, \quad y = \text{arc} \cot x,$$

$$y = \text{arc} \sec x, \quad y = \text{arc cosec}\, x.$$

These functions are called **inverse trigonometric** or **circular functions.**

We obtain their derivatives as follows. Differentiation of equation (1) gives

$$\frac{dx}{dy} = \cos y,$$

$$= \sqrt{1 - x^2},*$$

or by p. 142, $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}};$$

hence

(7) $$\frac{d}{dx} \text{arc} \sin x = \frac{1}{\sqrt{1 - x^2}}.$$

In the same way equation (3) yields

(8) $$\frac{d}{dx} \text{arc} \cos x = -\frac{1}{\sqrt{1 - x^2}}.$$

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* Formula 29, Appendix.

From the first of equations (5) we have

$$\frac{dx}{dy} = \frac{1}{\cos^2 y},$$

or, since $$\cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2},$$

we have $$\frac{dx}{dy} = 1 + x^2,$$

or $$\frac{dy}{dx} = \frac{1}{1 + x^2},$$

or, finally,

(9) $$\frac{d}{dx} \text{arc tan } x = \frac{1}{1 + x^2}.$$

Quite similarly we find from $x = \cot y$ (or $y = \text{arc cot } x$) that

(10) $$\frac{d}{dx} \text{arc cot } x = -\frac{1}{1 + x^2}.$$

EXERCISES XVI

Differentiate:

1. $x \log x$.
2. $\dfrac{\log x}{x}$.
3. $\log x \cdot e^x$.
4. $\dfrac{\log x}{e^x}$.
5. $(1 - x^2) \text{ arc sin } x$.
6. $(1 + x^2) \text{ arc tan } x$.
7. $e^x(\cos x - \sin x)$.
8. $e^x x^5$. *Ans.* $e^x x^4(x + 5)$.
9. $e^x(x^3 - 3x^2)$.
0. $e^x \text{ arc cot } x$.
11. $\dfrac{\text{arc cos } x}{1 - x^2}$.
12. $\dfrac{\text{arc tan } x}{2e^x}$.
13. $\text{arc tan } x + \text{arc cot } x$.
14. $\log x \cdot \sin x$.
15. $\dfrac{\cos x}{\log x}$.
16. $\tan x \cdot \log x$.
17. $\dfrac{\sin x}{x^4} + \log x$.
18. $\dfrac{1}{\log x} + 3 \log x$.

ART. **15. Differentiation of functions of functions.** Our previous formulæ enable us to perform the differentiation of a large variety of expressions, but they do not suffice to enable us to find the derivative of such simple functions as

$$(1) \qquad [(a^2 + x^2)]^3, \ \sin(x - a), \ \log\frac{a - x}{b - x},$$

and still precisely such functions as these occur much more frequently than do $\sin x$, x^n, or $\log x$. To treat these functions, we need therefore a more general process.

Each of the functions (1) is a function of a function of x, the first is a power of $a^2 + x^2$, the second is the sine of a difference, the third is the logarithm of a quotient. They have all, therefore, the form

$$(2) \qquad y = F(u),$$

where u is itself some function of x, which we denote by

$$(3) \qquad u = \phi(x),$$

so that we may write y in the form

$$(4) \qquad y = F[\phi(x)].$$

From equation (2), regarding y as a function of u, we have

$$(5) \qquad \lim_{h_1 \doteq 0} \frac{F(u + h_1) - F(u)}{h_1} = \frac{dy}{du}.$$

This is, by definition, the derivative of y when u is regarded as the independent variable (or, more briefly, the derivative of y with respect to u), provided h_1 is any quantity which may approach the limit zero.

From equation (3), we have, similarly,

$$\lim_{h_2 \doteq 0} \frac{\phi(x+h_2)-\phi(x)}{h_2} = \frac{du}{dx}. \tag{6}$$

From equation (4), we have

$$\lim_{h_3 \doteq 0} \frac{F[\phi(x+h_3)]-F[\phi(x)]}{h_3} = \frac{dy}{dx}. \tag{7}$$

Each of the equations (5), (6), and (7) defines the derivative indicated no matter what expressions are used as h_1, h_2, h_3, provided only that they can each be made to approach the limit zero.

We now avail ourselves of this fact to make a special choice of the quantities h_1, h_2, h_3, *viz.*

$$h_2 = h_3,$$

$$h_1 = \phi(x+h_3)-\phi(x).$$

It is apparent that this choice is permissible, because when h_3 approaches zero, h_1 and h_2 evidently approach zero.*

We have then from (5)

$$\frac{dy}{du} = \lim_{h_3 \doteq 0} \frac{F[u+\overline{\phi(x+h_3)-\phi(x)}]-F(u)}{\phi(x+h_3)-\phi(x)},$$

or replacing u by its value $\phi(x)$, we have

$$\frac{dy}{du} = \lim_{h_3 \doteq 0} \frac{F[\phi(x+h_3)]-F[\phi(x)]}{\phi(x+h_3)-\phi(x)}. \tag{8}$$

$$\frac{du}{dx} = \lim_{h_3 \doteq 0} \frac{\phi(x+h_3)-\phi(x)}{h_3}. \tag{9}$$

* The dependence of our results upon the tacit hypothesis which we always make, that our functions are all *continuous* functions (p. 160), is here very clearly seen.

But we have identically, for all values of h_3 however small,

$$(10)\quad \frac{F[\phi(x+h_3)]-F[\phi(x)]}{\phi(x+h_3)-\phi(x)}\cdot\frac{\phi(x+h_3)-\phi(x)}{h_3}$$
$$=\frac{F[\phi(x+h_3)]-F[\phi(x)]}{h_3}$$

Taking the limit as $h \doteq 0$, of both members of equation (10), we have, comparing with equations (8), (9), and (7),

$$(11)\qquad \frac{dy}{du}\cdot\frac{du}{dx}=\frac{dy}{dx}.$$

We have thus the important result:

If y is a function of u, and u in turn is a function of x, then the derivative of y with respect to x is equal to the derivative of y with respect to u, multiplied by the derivative of u with respect to x.

We proceed at once to apply this result, by finding the derivatives of the functions given in (1).

EXAMPLES

1. Let $\qquad y = \{a^2 + x^2\}^3.$

Put $\qquad u = a^2 + x^2,$

and accordingly, $\qquad y = u^3.$

By our previous results $\qquad \frac{du}{dx} = 2x;$

$$\frac{dy}{du} = 3u^2.$$

Hence
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= 3u^2 \cdot 2x$$
$$= 6x(a^2 + x^2)^2.$$

2. Let $y = \sin(x - a)$.

Put $u = x - a$.

Then $y = \sin u$.

$$\frac{du}{dx} = 1;$$

$$\frac{dy}{du} = \cos u.$$

Therefore
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \cos u = \cos(x - a).$$

3.
$$y = \log \frac{a - x}{b - x}.$$

Put
$$u = \frac{a - x}{b - x}.$$

Then $y = \log u$.

$$\frac{du}{dx} = \frac{(b - x)(-1) - (a - x)(-1)}{(b - x)^2}$$
$$= \frac{a - b}{(b - x)^2},$$

and
$$\frac{dy}{du} = \frac{1}{u}.$$

Hence,
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{a - b}{(b - x)^2}$$
$$= \frac{b - x}{a - x} \cdot \frac{a - b}{(b - x)^2}$$
$$= \frac{a - b}{(a - x)(b - x)}.$$

4. To find the derivative of a^x.

Put $y = a^x$.

We have $a = e^{\log a}$;

$$a^x = (e^{\log a})^x = e^{x \log a}.$$

Put $x \log a = u$. Hence $y = e^u$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u \cdot \log a.$$

Therefore
$$\frac{da^x}{dx} = a^x \log a.$$

The work of computation in examples like the preceding can be somewhat abbreviated by not formally introducing the function u. Thus in the first example we may write at once,

$$\frac{dy}{dx} = 3(a^2 + x^2)^2 \frac{d(a^2 + x^2)}{dx},$$

where all that remains to be done is to differentiate the second factor, giving

$$\frac{dy}{dx} = 3(a^2 + x^2)^2 \cdot 2\,x = 6\,x(a^2 + x^2)^2.$$

Similarly, the other examples can be worked without the formal introduction of u, and as the student becomes familiar with the method of this section by practice, he will find that the explicit use of the function u may gradually be omitted. But the beginner is earnestly advised *always* to use the auxiliary function u until he has acquired thorough control of the practical application of the method. Even then he should make formal use of the auxiliary functions whenever the expression to be differentiated is complicated, as confusion and errors may otherwise easily arise.

The results of this section can be extended immediately to functions of functions of functions, and so on. In such cases several auxiliary functions are necessary and their formal use is imperative.

EXERCISES XVII

1. $y = (4\,x - 9)^5$.

2. $y = (3\,x^3 - 5)^7$.

3. $y = \sin 5\,x$.

4. $y = \cos(2\,x^2 - 3\,x)$.

5. $y = (ax + b)^8$.

6. $y = \log 2\,x$.

7. $y = \log x^2$.

8. $y = \log^2 x$.

9. $y = \log \dfrac{x}{x+1}$.

10. $y = e^{8x}$.

11. $y = e^{2x+3}$.

12. $y = e^{5x^3}$.

13. $y = e^{2x^5 - 5x^2}$.

14. $y = \log \sin x$.

15. $y = \sin x^4$.

16. $y = \sin^4 x$.

17. $y = \sin 4\,x$.

18. $y = \sin^3 x \cos^5 x$.

19. $y = \tan x^3$.

20. $y = e^{\sin x}$.

21. $y = e^{\tan x - 1}$.

22. $y = \text{arc} \sin x^3$.

23. $y = (\text{arc} \sin x)^3$.

24. $y = \text{arc} \sin e^x$.

25. $y = \tan \dfrac{ax+b}{ax-b}$.

ART. **16. The derivative of a power with any exponent.** Considering x^n (n being a positive integer), we have shown that $\dfrac{d}{dx}x^n = nx^{n-1}$; and we shall now show that $\dfrac{dx^n}{dx} = nx^{n-1}$, even when n is a positive fraction.

Let $$y = x^n,$$

and let $$n = \frac{p}{q},$$

where p and q are positive integers; accordingly

(1) $$y = x^{\frac{p}{q}}.$$

Raising (1) to the qth power, we have

$$y^q = x^p.$$

Put (2) $$y^q = u,$$

then (p. 152), $$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$$

(3) $$= qy^{q-1}\frac{dy}{dx}.$$

From (2)

(4) $$\frac{du}{dx} = \frac{d(x^p)}{dx} = px^{p-1}.$$

Equating (3) and (4), and solving for $\dfrac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$

$$= \frac{p}{q}x^{p-1} \cdot (x^{\frac{p}{q}})^{-q+1}, \text{ by (1)}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}.$$

Therefore, replacing $\frac{p}{q}$ by its equal n,

(5) $$\frac{d}{dx}x^n = nx^{n-1}.$$

The formula for the derivative of a power holds, therefore, as well when n is a positive fraction as when it is a positive integer. We show further that it holds when n is a negative integer or a negative fraction.

Consider $y = x^n$, where $n = -r$, and r is a positive integer or fraction.

Then $$y = \frac{1}{x^r}.$$

Applying the rule for the differentiation of a fraction, we have

$$\frac{dy}{dx} = -rx^{-r-1}.$$

Replacing $-r$ by its equal n, and y by x^n,

(6) $$\frac{dx^n}{dx} = nx^{n-1}.$$

We thus have the general result that the formula just written is true for *all integral and fractional values of n.*

In particular, we have

(7) $$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}.$$

(8) $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

These formulæ are frequently applied.

EXAMPLES

1. $\frac{d}{dx}\sqrt[3]{x^2} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$,

2. $\frac{d}{dx}\left(\frac{a}{x}\right) = \frac{d}{dx}(ax^{-1}) = -ax^{-2} = -\frac{a}{x^2}$,

3. $\dfrac{d}{dx}\left(\dfrac{a}{x^n}\right) = \dfrac{d}{dx}(ax^{-n}) = -nax^{-n-1} = -\dfrac{na}{x^{n+1}}$,

4. $\dfrac{d}{dx}\left(\dfrac{a}{\sqrt{x}}\right) = \dfrac{d}{dx}(ax^{-\frac{1}{2}}) = -\tfrac{1}{2}\,ax^{-\frac{3}{2}} = \dfrac{-a}{2\sqrt{x^3}}$.

ART. **17. Logarithmic Differentiation.** In cases where the variable appears in the exponent as well as in the expression affected by the exponent, it is usually best to pass to logarithms before differentiating. A few examples will illustrate the method, which is known as **logarithmic differentiation.**

EXAMPLES

1. $$y = x^x.$$

Passing to logarithms,

$$\log y = x \log x.$$

Taking derivatives of both members with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x,$$

$$\frac{dy}{dx} = x^x(1 + \log x).$$

2. $$y = x^{e^x},$$

$$\log y = e^x \log x,$$

$$\frac{1}{y}\frac{dy}{dx} = e^x \log x + \frac{e^x}{x},$$

$$\frac{dy}{dx} = x^{e^x} e^x\left(\log x + \frac{1}{x}\right).$$

3. $$y = \sqrt{(ax)^{\log bx}},$$

$$\log y = \tfrac{1}{2} \log bx \log ax,$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\frac{b}{bx}\log ax + \frac{a}{ax}\log bx\right\}$$

$$= \frac{1}{2x}\left\{\log ax + \log bx\right\}$$

$$= \frac{1}{2x}\log abx^2,$$

$$\frac{dy}{dx} = \frac{\sqrt{(ax)^{\log bx}}\log abx^2}{2x}.$$

ART. 18. **Summary of Results.** We collect into a table, the principal results which we have established.

TABLE OF RESULTS

$$\frac{d}{dx}x^n = nx^{n-1},\text{*} \qquad \frac{d}{dx}\log x = \frac{1}{x},$$

$$\frac{d}{dx}c = 0, \qquad \frac{d}{dx}\log_a x = \frac{1}{x}\log_a e,$$

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}e^x = e^x,$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}a^x = a^x \log a,$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}, \qquad \frac{d}{dx}\text{arc}\sin x = \frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}, \qquad \frac{d}{dx}\text{arc}\cos x = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}\sec x = \tan x \sec x, \qquad \frac{d}{dx}\text{arc}\tan x = \frac{1}{1+x^2},$$

$$\frac{d}{dx}\text{cosec}\, x = -\cot x \,\text{cosec}\, x, \qquad \frac{d}{dx}\text{arc}\cot x = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} + \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

If $y = f(u)$ and $u = \phi(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Slope of tangent = Value of Derivative.

* The constant n may be positive, negative, integral, or fractional.

EXERCISES XVIII (MISCELLANEOUS)

Differentiate:

1. $y = x^3$.
2. $y = 5x^2$.
3. $y = x^{\frac{1}{2}}$.
4. $y = ax^5 + bx^6$.
5. $y = a + bx + cx^2$.
6. $y = x^{\frac{2}{3}}$.
7. $y = \sqrt[5]{x^6}$.
8. $y = \dfrac{1}{x}$.
9. $y = 5x^{-5}$.
10. $y = \sin^2 x$.
11. $y = \sqrt{\sin x}$.
12. $y = x \cos x$.
13. $y = x \log x$.
14. $y = x^2 \tan x + \dfrac{x}{\cos x}$.
15. $y = \sqrt{1 - x^2}$.
16. $y = (x + 1)(x + 2)$.
17. $y = (2x^2 - 4)(3x + 5)$.
18. $y = x^5 e^x$.
19. $y = \sin^m x \cos^r x$.
20. $y = \cos \log x$.
21. $y = \log \sin^n x$.
22. $y = e^{nx^r}$.
23. $y = e^{e^x}$.
24. $y = \sec(3x + 5)$.
25. $y = (x^2 - 14x + 2)^8$.
26. $y = x^2\sqrt{1 + x^2}$.
27. $y = \log(1 + x^2)$
28. $y = \log \dfrac{1 + x}{1 - x}$.
29. $y = x^n + nx^{n-1}$.
30. $y = \log(\log x)$.
31. $y = \log \sin(ax + b)$.
32. $y = \log \tan \dfrac{x^2}{x + 3}$.
33. $y = (x + a)(x + b)(x + c)$.
34. $y = (x-1)(x-2)(x-3)(x-4)$.
35. $y = (x - a_1)(x - a_2)\cdots(x - a_n)$.
36. $y = e^{-\sin^2 x^2}$.
37. $y = e^{\arctan x}$.
38. $y = e^{\log \frac{x^2}{5}}$.
39. $y = e^{ax}(a \sin x - \cos x)$.
40. $y = \log(x + \sqrt{1 + x^2})$.
41. $y = e^{\frac{1}{x}}$.
42. $y = e^x \cdot x^n$.
43. $y = \tan x + \frac{1}{3} \tan^3 x$.
44. $y = \tan x - \cot x - 2$.
45. $y = \log\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$.
46. $y = (\log x^2)^2$.
47. $y = \arcsin \dfrac{1 - x^2}{1 + x^2}$.
48. $y = \log\sqrt{\dfrac{a \cos x - b \sin x}{a \cos x + b \sin x}}$. *Ans.* $\dfrac{-ab}{a^2 \cos^2 x - b^2 \sin^2 x}$.
49. $y = x^{x^n}$. *Ans.* $x^{x^n + n - 1}(n \log x + 1)$.
50. $y = \log\sqrt{\dfrac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2} - x}}$. *Ans.* $\dfrac{1}{\sqrt{1 + x^2}}$.
51. $y = (\sin x)^x$. *Ans.* $(\sin x)^x\{\log \sin x + x \cot x\}$.
52. $y = \left(\dfrac{a}{x}\right)^x$. *Ans.* $\left(\dfrac{a}{x}\right)^x(\log a - \log x - 1)$.

53. $y = \left(\frac{1-x}{x}\right)^{\left(\frac{1-x}{x}\right)}$.

HINT. — Put $\frac{1-x}{x} = z$. *Ans.* $\frac{dy}{dx} = -\frac{1}{x^2}\left(\frac{1-x}{x}\right)^{\left(\frac{1-x}{x}\right)}\left[1 + \log\frac{1-x}{x}\right]$.

54. $y = \sqrt[x]{x}$. *Ans.* $\frac{dy}{dx} = \frac{\sqrt[x]{x}}{x^2}(1 - \log x)$.

55. $y = \frac{2x^4}{(9x-13)^3\sqrt{9x^2 - 13x}}$. *Ans.* $\frac{dy}{dx} = -\frac{91x^3}{(9x-13)^4\sqrt{9x^2-13x}}$.

56. $y = \frac{108 - 18\sqrt{x} - 3x - \frac{5x}{6}\sqrt{x}}{\sqrt[3]{2 - \sqrt{x}}}$. *Ans.* $\frac{dy}{dx} = \frac{10x}{9\sqrt[3]{(2-\sqrt{x})^4}}$.

57. $y = \frac{\frac{3}{x^3} + 20x^3 + \frac{200}{9}x^9}{\sqrt{(3+5x^3)^3}}$. *Ans.* $\frac{dy}{dx} = -\frac{27}{x^4\sqrt{(3+5x^3)^5}}$.

58. $y = \frac{1}{5\sqrt[3]{\left(1+\frac{1}{x^3}\right)^5}} - \frac{1}{8\sqrt[3]{\left(1+\frac{1}{x^3}\right)^8}}$. *Ans.* $\frac{dy}{dx} = \frac{x^4}{\sqrt[3]{(x^3+1)^{11}}}$.

ART. **19. Continuity and discontinuity.** We reached the notion of the derivative by taking up the problem, to determine the position of the tangent of a curve, the speed of a moving point, or the coefficient of expansion of a metal rod. In each of these cases we had under consideration the determination of a quantity with a definite geometric or physical signification. We then extended the method of computation of the derivative which we used in the instances just named to the more usual classes of functions which occur in mathematics. The examples which we have treated show that for all these functions the derivative exists and may be determined in a simple manner, and besides may frequently be brought into connection with some natural phenomenon.

We do not wish, however, to pass on without alluding to the fact that for certain functions of pure mathematics, as

well as in some applications, exceptions to our previous results may present themselves.

We have already called a function that is altered little at will when the independent variable undergoes a sufficiently slight change, a *continuous* function. (It is hardly necessary to remark that the course of the processes occurring in nature, whether physical phenomena or chemical reactions, can usually be represented by continuous functions. *Natura non facit saltus.*)

We say of a curve whose equation is

$$v = f(t)$$

and which has a break in its course, like that in Fig. 42, that it is **discontinuous** for the value $t = OQ$, and that it has a **discontinuity** at this point. The same expressions are also applied to the function $f(t)$ itself. If we let t increase by an increment, as small as we please, added to the value $t = OQ$, $f(t)$ does not change (as did all the functions hitherto considered) by an increment which is also very small, but by an increment at least equal to PP', no matter how small the increment of t may be taken. The derivative has two values (usually different) at the point $t = OQ$ which correspond to the positions of the tangents (Fig. 42) which can be drawn to the curve at the distinct points P and P'.

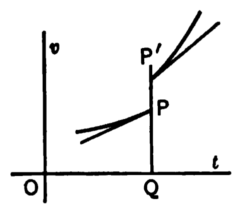

Fig. 42.

We also say that a function is *discontinuous* for a certain value of x, when for that value of x the function becomes infinite. Such a function is, for example,

$$y = \frac{1}{a - x}.$$

As $x \doteq a$, y becomes large without bound; the function y is therefore said to be discontinuous at the point $x = a$. The derivative of this function has the value,

$$\frac{dy}{dx} = \frac{1}{(a-x)^2},$$

and is likewise infinite for the value $x = a$. We have met several such functions in the preceding chapters. To select them, we have only to examine the graph and see whether the curve contains branches in which the ordinate becomes infinite for a finite value of x. Such curves are, for example, those for $\log x$ (p. 141), for Boyle's Law (p. 4), and others.

We add a simple example of a function which becomes discontinuous without becoming infinite. For brevity, we introduce temporarily the notation I_x to denote the greatest integer contained in the value of x taken positively. Thus, $I_{4.5} = 4$, and $I_{-6.7} = 6$.

Considering now the function

$$y = x + I_x,$$

we see that when $-1 < x < 1$,

$$y = x + 0,$$

but when $1 \overline{\gtrless} x < 2$,

$$y = x + 1,$$

and when $2 \overline{\gtrless} x < 3$,

$$y = x + 2, \text{ etc., etc.}$$

When x approaches 1, y also approaches 1; *i.e.*,

$$\lim_{x \doteq 1} [x + I_x] = 1,$$

but when $x = 1$, $y = 2$. When $x = 1$, any diminution in the value of x, no matter how slight, causes the value of y to diminish by more than a unit; *i.e.* y is discontinuous at the point $x = 1$. Likewise, y is discontinuous for all integral

values of x, except zero. The graph for this function (Fig. 43) has a break at each integral value of the abscissa.

A second illustration is the function $y = I_x$. We give its graph (Fig. 44), and leave the detailed discussion as an exercise for the reader.

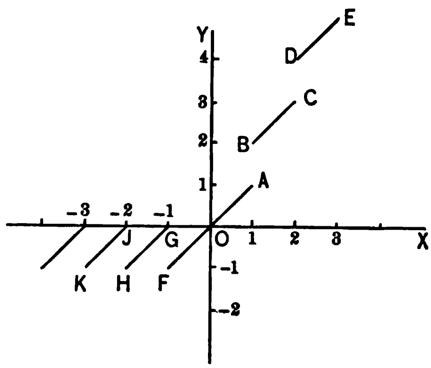

FIG. 43.

In other cases, curves may indeed be free from breaks or discontinuities, and yet at some point have a *sudden* change of direction. In this case, the curve for the derivative has a discontinuity at that point. A good example of this is the curve for the vapor-tension* of a substance. The vapor tension p of a solid increases gradually under rising temperature; as soon as the melting point is reached and the solid assumes the liquid condition, the vapor-tension *instantly* begins to increase at a more rapid rate, and the vapor-tension curve accordingly has a sudden change of direction, as is illus-

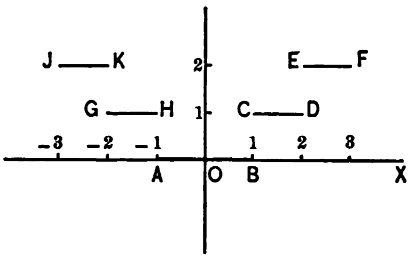

FIG. 44.

* Every solid and liquid has a tendency to evaporate, and when it is confined in a limited space, the vapor produced exerts a tension which can be measured and is known as the **vapor-tension** of the substance.

trated graphically in Fig. 45. The vapor-tension itself has no discontinuity at the melting point, for at that temperature the value of the vapor-tension is just the same for the liquid as for the solid body. But the derivative is discontinuous there; for if we raise the melting temperature $t = OQ$, by an increment Δt, as small as we please, the derivative passes from the value of the tangent of the angle α to the tangent of the angle α'; the change in magnitude of the derivative does not approach zero, no matter how small the increment of temperature is taken.

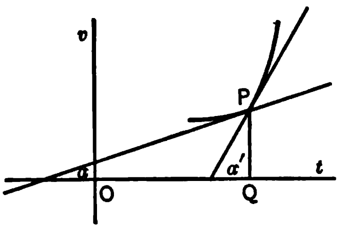

FIG. 45.

We conclude with a formal analytic definition of continuity, embodying in mathematical symbols the ideas we have just set forth.

A function $f(x)$ is continuous for the value $x = a$, if

$$\lim_{h \doteq 0} [f(a+h) - f(a)] = 0.$$

The function $f(x)$ is always continuous if we have, irrespective of the value of x,

$$\lim_{h \doteq 0} [f(x+h) - f(x)] = 0.$$

In a continuous function this limit is zero, both (1) when h approaches zero through positive values, and (2) when h approaches zero through negative values.

In a discontinuous function, on the other hand, this limit is different from zero in at least one of these cases.

Thus, in the function $f(x) = x + I_x$, discussed above,

$$\lim_{h \doteq 0} [f(a+h) - f(a)]$$

(a being a positive integer) is zero in case (1), but unity in case (2). If a is a negative integer, the limit is -1 in case (1) and zero in case (2). If a is zero, or not an integer, the limit is zero in both cases, and the function is continuous for these values of a.

In what is to follow, just as in what has preceded, we shall, as a rule, pay no attention to the possibility of exceptional values of x, for which the function is not continuous. The results which we deduce in this manner are, of course, only proved to hold for continuous functions, but we shall usually apply our results to functions of such simple nature that we shall take their continuity for granted.

CHAPTER V

THE FUNDAMENTAL CONCEPTIONS OF THE INTEGRAL CALCULUS

ART. 1. **The problems of the Integral Calculus.** We learned in Chapter III. that, in the theoretic study of natural phenomena, two essentially different problems confront us: one assumes the laws to be given and endeavors to find out what the condition of the process is at any moment; the other requires the deduction of the law controlling the entire process from the facts relating to the single phases of it; the one problem is the inverse of the other. The former problem led us to the conceptions and methods of the Differential Calculus. We now take up the inverse problem, and begin with the following illustrative example concerning the motion of a freely falling body:

The value of v at any instant being given by the equation

(1) $$v = gt,$$

to derive the formula for the space l traversed in the interval of time t, viz.

(2) $$l = \tfrac{1}{2} gt^2.$$

We might start out from the equation

$$v = \frac{dl}{dt},$$

which we have already established (p. 109). But we proceed in a somewhat more detailed manner in order to make per-

fectly clear the general nature of the process for use in other problems.

Formula (1) was obtained by means of an auxiliary point which passes at a uniform rate over the space $l_1 - l$ during the interval of time $t_1 - t = \tau$, and whose velocity by definition is

$$(3) \qquad V = \frac{l_1 - l}{\tau}.$$

If now we take the limit of both members of (3) as τ approaches zero, we have, as the limit of V, the velocity v at P_1; but the limit of the right member is what we have called the derivative, and we have introduced a notation according to which the limit of the right member is denoted by $\frac{dl}{dt}$. As its value proves to be gt, we have finally

$$(4) \qquad v = \frac{dl}{dt} = gt.$$

The specifically physical part of the problem concludes with the establishing of equation (4). To pass from (4) to (2) is a matter of pure calculation alone: a function l is to be found whose derivative is known. *This is the inverse* of the problems treated in Chapters III. and IV. There we had to determine the derivatives of given functions; here the derivative is given, and the function of which it is the derivative is to be determined. In the example under discussion, it is to be shown that $l = \frac{1}{2}gt^2$ is the function of which gt is the derivative.

As a second example we take the inversion of sugar (*i.e.* decomposition of cane sugar in aqueous solution into dextrose and levulose through presence of acids). By **speed of reaction** we mean the rate at which the sugar is inverted. If as much sugar is added to the solution as

disappears on account of the inversion, equal quantities of sugar will be inverted in equal intervals of time; the reaction takes place with constant speed. The so-called **Law of Mass Action** states with reference to the process of sugar inversion that the mass of sugar thus inverted in the unit of time is directly proportional to the mass of sugar still unchanged.

Let a represent the mass of sugar present in the solution at the beginning of the inversion, and let x be the mass of sugar inverted in the time t; then the mass of sugar at the close of the time t is $a - x$. Let the mass of sugar inverted in the interval of time Δt immediately following be denoted by Δx; in accordance with our method, the inversion is to be regarded as a perfectly *uniform* process during the time Δt. According to the Law of Mass Action, the amount inverted is proportional to the mass of sugar remaining unaltered in the solution; if its value for the unit-mass of sugar be denoted by k, then its value for the mass of sugar $a - x$ is $k(a - x)$, which expression gives the mass of sugar inverted uniformly in the unit of time.

If $a - x$ be the amount of sugar left at the beginning of the interval Δt, and $a - x_1$ that at its end, then $a - x > a - x_1$, and the smaller the interval Δt, the more nearly the quantities $a - x$ and $a - x_1$ are equal. The amount of sugar inverted in the time Δt, if the mass of sugar remains constant as at the beginning of the interval, would be $k(a - x)\Delta t$. But this is too great, as the mass of sugar is decreasing. If the mass of sugar were constantly the same as at the end of the interval, the amount inverted would be $k(a - x_1)\Delta t$. But this is too small, as the mass at the end is the smallest mass in the interval. The true mass inverted is then less than the former and greater than the

latter mass. If we denote it by $km\Delta t$, then m is less than $a-x$ and greater than $a-x_1$. Letting $m = a - x - \epsilon$, where $0 < \epsilon < x_1 - x$, the true mass inverted is denoted by

$$k(a - x - \epsilon)\Delta t.$$

But we have previously called this same amount of sugar Δx, so that

$$\Delta x = k(a - x - \epsilon)\Delta t,$$

or

$$\frac{\Delta t}{\Delta x} = \frac{1}{k(a - x - \epsilon)}.$$

Taking the limit, as Δx approaches zero (then x approaches x_1, and ϵ also approaches zero), we have

$$\frac{dt}{dx} = \frac{1}{k(a-x)}.$$

The chemical law has thus been brought into a mathematical form. We have now to determine what functional relation exists between x and t that finds its expression in the last equation; in other words, what function of x has $\dfrac{1}{k(a-x)}$ as its derivative. It is easily verified by differentiation that

$$t = -\frac{1}{k}\log(a - x)$$

has the given derivative,* and this expression is the functional relationship between x and t; it may also be written

$$t = \frac{1}{k}\log\frac{1}{a-x}.$$

ART. 2. **Integrals.** The preceding illustrations lead us to the general problem of the determination of a function

* A fuller discussion of this step will be given on pp. 181–182 and 189–190.

whose derivative is known, the inverse problem to that dealt with in the Differential Calculus, where it was required to find the derivative when the function was given.

The branch of Mathematics that deals with the finding of functions whose derivatives are known is called **Integral Calculus.** This name will immediately seem more apt if we note that the problems treated in the Integral Calculus seek to find the laws that regulate the entire (*integer*) course of some variation.

Let x^n be the derivative of a certain function y; then

$$\text{(1)} \qquad \frac{dy}{dx} = x^n.$$

Or, more generally, let the derivative be denoted by $f(x)$ and the required function by $F(x)$; then

$$\text{(2)} \qquad \frac{dF(x)}{dx} = f(x).$$

Any function $F(x)$ that satisfies this equation may be symbolically represented by

$$\text{(3)} \qquad F(x) = \int f(x)\,dx;$$

this is called the **integral** of $f(x)$.

The process by which (3) is derived from (2) is called **integration.**

According to the above notation, we understand by $\int f(x)\,dx$, a function whose derivative is $f(x)$. This is simply a convention. Instead of writing "the function whose derivative is $f(x)$," we write $\int f(x)\,dx$. We might just as well introduce the notation $If(x)$, or $\{f(x)\}$, or any other which we might choose. I, $\{\ \}$, $\int \cdots dx$, would alike be symbols meaning "the function whose derivative

is." In particular, dx is a part of the symbol, just as much as the second brace would be if we had introduced the symbol $\{\,\}$. The dx has no meaning in itself, but taken with the $\int$, it means "the function whose derivative with respect to x is." It might seem that the sign $\int$ would alone be a sufficient symbol, and ordinarily it would be, but the dx is retained as part of the symbol from historical reasons, since the notation is firmly imbedded in a large mass of mathematical literature, and also because it is often necessary to indicate what is to be regarded as the variable quantity in forming the derivative in question, and this is conveniently done by the x occurring in the dx. While the symbol $\int \cdots dx$ has the *meaning* just indicated, it is usually *read*, as we have already mentioned above, "the integral of" whatever function may occur in the symbol, and if it be necessary to specify what is to be regarded as the variable, the symbol is read "the integral with respect to x of."

We have said that dx is to be regarded as a part of the symbol, yet it is not arbitrarily fixed so. It is rather the *trace* of a finite quantity which was made to approach the limit zero in the earlier history of the function under consideration. We have seen this illustrated in the first paragraph of this chapter, and shall see it still more clearly when we come, in Chapter VIII., to a second definition of an integral which is there developed.

On applying this notation to the examples considered on pp. 166–169, we obtain

$$l = \int gt\, dt = \frac{1}{2}gt^2,$$

and

$$t = \int \frac{1}{k}\frac{1}{a-x}\, dx = \frac{1}{k}\log\frac{1}{a-x}.$$

In the Differential Calculus we grew accustomed not to keep $\frac{d}{dx}$, the symbol for the operation of differentiation, invariably separated sharply from the quantity to which the operation was to be applied, but sometimes to write for compactness $\frac{dy}{dx}$, $\frac{d(1+x^2)}{dx}$, etc., instead of $\frac{d}{dx}y$, $\frac{d}{dx}(1+x^2)$, etc.

The notation for integrals is often made a little more compact in a similar manner. Thus, instead of $\int\frac{1}{x}\,dx$, $\int 1\,dx$, etc., we frequently write $\int\frac{dx}{x}$, $\int dx$, etc., unity being omitted in the latter in accordance with the general custom of omitting unity when no confusion is caused by doing so. In every case $\int\cdots dx$ constitutes the symbol of integration, and the remainder of what is written, the function to be integrated.

ART. 3. **The integral calculus as an inverse problem.** The question now arises: How can the integral $F(x)$ of a given function $f(x)$ be determined? Before taking up this question we note that the operations of the Differential and the Integral Calculus are opposite in character.

The contrast between the Integral and the Differential Calculus is analogous to that between multiplication and division, or between involution and evolution; and in each of these cases the inverse operation is essentially more difficult than the direct. While we secured in the Differential Calculus a general method for finding the derivative when the function is given, there exists no corresponding general method for the inverse problem; each case requires special treatment.

From what has been stated above, $F(x)$ or $\int f(x)\,dx$

is to be understood as a function whose derivative is $f(x)$, as shown in equation (2), p. 170. If we substitute in this equation for $F(x)$ its value given in equation (3), p. 170, we have

$$(1)\qquad \frac{d}{dx}\int f(x)\,dx = f(x),$$

an equation which proves that the operations of differentiation denoted by $\frac{d}{dx}$ and of integration denoted by $\int \cdots dx$ counteract each other, just as is the case with the operations of extracting roots and raising to powers.

In a similar manner we obtain

$$(2)\qquad \int\left[\frac{d}{dx}F(x)\right]dx = F(x),$$

proving again that $\frac{d}{dx}$ and $\int \cdots dx$ have contrary effects.

If, for brevity, we put $u = F(x)$, equation (1) takes the simple form

$$(3)\qquad \int\frac{du}{dx}\cdot dx = u,$$

an equation which we shall have occasion to use repeatedly.

Equations (1) and (2) show that the derivative of an integral as well as the integral of a derivative always gives the original function, just as the nth root of an nth power, or the nth power of an nth root, always gives the fundamental number. For every two inverse kinds of calculation there exist two equations of this sort. If we take the raising to powers as corresponding to differentiation, and the extraction of roots as corresponding to integration, the analogue of the first equation is

$$(\sqrt[n]{a})^n = a,$$

where the root of a is first taken and the result is raised to a power. The analogue of the second equation is

$$\sqrt[n]{(a^n)} = a,$$

where a is first raised to a power and the root of the result taken. Likewise in (1) above, $f(x)$ is first integrated and the result differentiated, while in (2), $F(x)$ is first differentiated and then the result integrated.

ART. 4. **The constant of integration.** If $F(x)$ be an integral of $f(x)$ so that it satisfies the equation

$$\frac{dF(x)}{dx} = f(x),$$

it is clear that

(1) $$F(x) + C,$$

where C denotes a constant quantity, also has this property; for (p. 128)

$$\frac{d[F(x) + C]}{dx} = \frac{dF(x)}{dx} = f(x).$$

The function $F(x) + C$ can therefore be also regarded as an integral of $f(x)$; in other words, *a given function $f(x)$ has a boundless number of integrals;* every value of C determines one of them.

This may be expressed by the formula

(2) $$\int f(x)dx = F(x) + C.$$

There is no danger, however, that we shall run into error by usually hereafter writing our equations as we have thus far done, simply

$$\int f(x)dx = F(x);$$

for the exact reading of this equation is: *one* integral function of $f(x)$ is $F(x)$, while that of equation (2) is: *all*

functions of the form $F(x)+C$ are integral functions of $f(x)$. The constant C is called the **constant of integration.**

ART. 5. **The fundamental formulæ of the Integral Calculus.** Inasmuch as the Integral Calculus is the inverse of the Differential Calculus, a formula of the Integral Calculus may be derived from any formula of the Differential Calculus. Thus, if the equation

$$\frac{dF(x)}{dx}=f(x)$$

be considered, it is clear that the equation

$$\int f(x)dx = F(x)$$

may be at once deduced from it.

In this way a first set of integral formulæ may be obtained immediately from the formulæ established in Chap. IV. Thus, for the power x^{n+1} we have

$$\frac{d(x^{n+1})}{dx}=(n+1)x^n,$$

or

$$\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx}=x^n,$$

or, finally,

$$\int x^n dx=\frac{x^{n+1}}{n+1}.$$

Similarly, the formula

$$\frac{d(-\cos x)}{dx}=\sin x$$

may be expressed in another form as

$$\int \sin x\, dx = -\cos x.$$

In this wise we obtain the following preliminary

TABLE OF INTEGRALS

1. $\int x^n dx = \frac{x^{n+1}}{n+1}.$

2. $\int \cos x\, dx = \sin x.$

3. $\int \sin x\, dx = -\cos x.$

4. $\int \frac{dx}{\cos^2 x} = \tan x.$

5. $\int \frac{dx}{\sin^2 x} = -\cot x.$

6. $\int e^x dx = e^x.$

7. $\int a^x dx = \frac{a^x}{\log a}.$

8. $\int \frac{dx}{x} = \log x.$*

9. $\int \frac{dx}{\sqrt{1-x^2}} = \text{arc} \sin x.$

10. $\int \frac{dx}{1+x^2} = \text{arc} \tan x = -\text{arc} \cot x.$†

ART. 6. **The geometric signification of the constant of integration.** There is but little in elementary mathematics analogous to the boundless number of integrals belonging to one and the same derivative, $f(x)$. Indeed, in extracting the square root of a quantity we obtain two solutions, and a cube root has three values, but nowhere does there occur as

* The integral on the left side is formally contained in the first integral of the Table. But in that case the first integral formula refuses to work, since $n + 1 = 0$; the above formula is therefore substituted for it. This illustrates how the Integral Calculus may lead us to new functions. If we had not previously defined *logarithms* and studied their properties, in particular, their derivatives, we should not be able to find the integral denoted by $\int \frac{dx}{x}$; but the proposing of this very simple function for integration might lead us to study the function whose derivative is $\frac{1}{x}$, and thus to discover the fundamental properties of logarithms.

† This result means that arc $\tan x$ and $-$ arc $\cot x$ are both functions that have $\frac{1}{1+x^2}$ as derivative; that is, they are both integrals of $\frac{1}{1+x^2}$, and can therefore differ only by a constant; this we know from trigonometry to be the case.

here, in finding the integral function for the derivative $f(x)$, an unlimited number of results. The formal explanation of this point has been presented previously. We now proceed to discuss the geometric meaning of the constant of integration.

Let $F(x)$ be any integral of $f(x)$, so that

$$\frac{dF(x)}{dx} = f(x).$$

If $$y = F(x),$$

this equation represents a curve, for which (p. 117)

$$\tan \alpha = \frac{dy}{dx} = \frac{dF(x)}{dx} = f(x); \tag{1}$$

this equation determines the position of the tangent at any point in our curve. The determination of the integral of the given function $f(x)$ then, geometrically speaking, amounts to determining the ordinate y of a curve when the direction of the tangent at every one of its points is known.

It is easily seen that the problem of constructing a curve from a knowledge of its tangents leads to a countless number of curves. For we observe that the determination of the tangents by means of equation (1) is made so that the angle τ in any point of the curve depends only upon the abscissa x of that point. If any curve whatever be known that fulfils the conditions of the problem, and it be moved any given distance, parallel to itself, in the direction of the axis of y, it will in its new position also satisfy the conditions of the problem. For example, if A_1 (Fig. 46) be a point which reaches the position B_1, both A_1 and B_1 have the same abscissa, and their tangents have remained parallel throughout the motion.

We can show the same thing in another manner, which also furnishes us with a means of actually constructing the curves under discussion when equation (1) is given, and thus of determining the required integral graphically. Let A_1 (Fig. 46) be any point of the plane, and let its coördinates be x_1, y_1. We determine an angle τ_1 by means of the equation

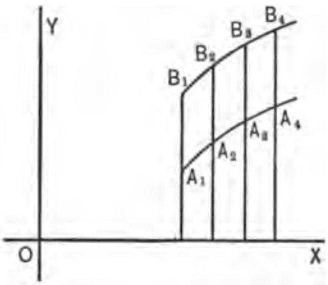

FIG. 46.

$$\tan \tau_1 = f(x_1),$$

and draw a straight line through A_1, making this angle with the x-axis. On this straight line we take, near to A_1, another point A_2 whose abscissa is x_2,* and calculate another angle τ_2 from the equation

$$\tan \tau_2 = f(x_2).$$

We now draw through A_2 a straight line with slope τ_2, and take upon it a point A_3 near to A_2; the abscissa of this point being x_3, the angle τ_3 is determined by the equation

$$\tan \tau_3 = f(x_3).$$

By continuing thus, we get a series of lines through A_1, A_2, A_3, $\cdots$, which form with the x-axis the same angles as do the tangents of the required curve. The nearer the points A_1, A_2, A_3, $\cdots$, are together, the closer the polygon approaches the curve, and its limit is the curve itself. In this way

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* It is simplest to choose the abscissæ x_1, x_2, x_3, $\cdots$, so that

$$x_2 - x_1 = x_3 - x_2 = \cdots;$$

for example, $x_1 = 5,\ x_2 = 5.1,\ x_3 = 5.2$, etc.

we have actually constructed a curve corresponding to equation (1).

Since we had perfect freedom in the choice of the starting point A_1 for our construction, the value of the ordinate y_1 belonging to the abscissa x_1 is arbitrary also. If we substitute for this value of y_1 another value, as Y_1, corresponding to the point B_1, we can in a similar manner draw the sides of a polygon B_1, B_2, B_3, $\cdots$, belonging to another curve also representing an integral. Since only the abscissæ x_1, x_2, x_3, $\cdots$ occur in the equations for τ_1, τ_2, τ_3, $\cdots$, we see immediately that the sides B_1B_2, B_2B_3, B_3B_4, $\cdots$ are parallel to the sides A_1A_2, A_2A_3, A_3A_4, $\cdots$. Therefore,

(2) $$A_1B_1 = A_2B_2 = A_3B_3 = \cdots,$$

or $$Y_1 - y_1 = Y_2 - y_2 = Y_3 - y_3 = \cdots;$$

if we denote by C the constant value of this difference, we obtain finally

$$Y_1 = y_1 + C,\ Y_2 = y_2 + C,\ Y_3 = y_3 + C \cdots;$$

that is,

(3) $$Y = y + C.$$

If the equation of the first curve be written in the form,

(4) $$y = F(x),$$

then the equation of the second curve becomes

(5) $$Y = F(x) + C,$$

and this is the equation we started out to explain.*

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* The above considerations show further that every integral function is deduced from $F(x)$ by the addition of a constant; on p. 174 we have proved only that all functions of the form $F(x) + C$ are integral functions.

We draw the conclusion, further, that in fixing upon any particular integral among the total number of possible ones, we are perfectly free to prescribe the value which the function shall have corresponding to any one given value of x.

ART. 7. **The physical signification of the constant of integration.** To illustrate the physical signification of the constant C, and to understand the necessity of its occurrence, we begin with the motion of freely falling bodies, for which

$$(1) \qquad \frac{dv}{dt} = g,$$

whence

$$(2) \qquad v = \int g\,dt,$$

or,
$$(3) \qquad v = gt + C.$$

We have already considered the motion of freely falling bodies (p. 167), and found there that $v = gt$. But we treated then only motion in which the falling body was at rest at the beginning of its fall, that is, at the moment $t = 0$. The laws of freely falling bodies embrace also the cases of motion in which, at the instant when gravitation commences to act, the body in question already possesses a certain velocity of its own, which may be directed upwards as well as downwards.

In order to single out any one of these motions, we must know what its velocity V is at some moment. It is advisable to take the moment when $t = 0$, since the initial velocity is usually given. Let this initial velocity be v_0, so that substituting $t = 0$ and $v = v_0$ in (3), we see that

$$(4) \qquad v_0 = C;$$

and for the motion in question we have then

(5) $$v = gt + v_0,$$

which determines the velocity v at any given moment t.

If the body be thrown vertically upward, its initial velocity is negative; we put

$$v_0 = -V,$$

and have $$v = gt - V.$$

The question may arise: when does the body come to rest, or change the sense of its motion? This occurs when v becomes equal to zero; the corresponding value of t is derived from the equation

$$0 = gt - V,$$

and is

(6) $$t = \frac{V}{g}.$$

We next take up the discussion of the inversion of sugar, for which we have already (p. 169) established the equation

(7) $$\frac{dt}{dx} = \frac{1}{k(a - x)},$$

which by integration (with the addition of the constant C) becomes

(8) $$t = \frac{1}{k}\log\frac{1}{a - x} + C.$$

C must naturally have a definite value for any given reaction, and this value can be determined as follows. If, as is customary, the time be counted from the moment when the reaction begins, the mass of sugar inverted at the time $t = 0$ is $x = 0$, and we have the equation

$$0 = \frac{1}{k}\log\frac{1}{a} + C,$$

which fixes the value of C. If we substitute this value of C in equation (8), we find

$$(9) \qquad t = \frac{1}{k}\log\frac{1}{a-x} - \frac{1}{k}\log\frac{1}{a}$$

$$= \frac{1}{k}\log\frac{a}{a-x}.^*$$

In practice, the constant C is generally determined in another way. A direct observation is made of the mass of sugar x_1 inverted in a time t_1. Then we have the equation

$$(10) \qquad t_1 = \frac{1}{k}\log\frac{1}{a-x_1} + C,$$

from which the value of C may be found. If this value of C be substituted in equation (8), it follows that

$$t_1 - t = \frac{1}{k}\log\frac{1}{a-x_1} - \frac{1}{k}\log\frac{1}{a-x}$$

$$= \frac{1}{k}\log\frac{a-x}{a-x_1}{}^*,$$

and finally

$$(11) \qquad k = \frac{1}{t_1 - t}\log\frac{a-x}{a-x_1}.$$

This is the best form that our equation can be made to assume for its experimental corroboration. It shows that its right-hand member must be a constant, and it is easy to find out whether or not this is the case by substituting various values of t, with the corresponding values of x, † as found by experiments.

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* Formula 6, Appendix.

† See the applications in Chapter VII., pp. 227, 244.

Exercise. Long (*Journal of the American Chemical Society*, Vol. XVIII, p. 129, 1896) found the following amounts of sugar x inverted in the times t:

$a = 43.91$

$t =$	30	60	120	180	300
$x =$	3.91	7.56	14.61	19.06	28.09

Compute the values of $\frac{1}{t_1 - t}\log\frac{a-x}{a-x_1}$.*

* The formula is true for Naperian logarithms only, but common logarithms may be used here, since we wish simply to verify that the value is constant.

CHAPTER VI

THE SIMPLER METHODS OF INTEGRATION

ART. 1. **Integration of sums and differences.** In Chapter III, p. 127, we saw that

$$(1) \qquad \frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx},$$

and by integrating this, we obtain

$$u+v=\int\left(\frac{du}{dx}+\frac{dv}{dx}\right)dx.$$

But (p. 173), $u=\int\frac{du}{dx}\,dx$, and $v=\int\frac{dv}{dx}\,dx$;

hence

$$(2) \qquad \int\left(\frac{du}{dx}+\frac{dv}{dx}\right)dx=\int\frac{du}{dx}\,dx+\int\frac{dv}{dx}\,dx;$$

in words, *the integral of a sum of two terms is equal to the sum of the integrals of the separate terms.*

A similar formula evidently holds for the sum of any given number of terms.

Likewise, by integrating the expression

$$(3) \qquad \frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx},$$

we find that

$$(4) \qquad \int\left(\frac{du}{dx}-\frac{dv}{dx}\right)dx=\int\frac{du}{dx}\,dx-\int\frac{dv}{dx}\,dx;$$

in words, *the integral of a difference is equal to the difference of the integrals.*

Finally, we obtain by integrating the formula

$$(5) \qquad \frac{d}{dx}(au) = a\frac{du}{dx},$$

in which a denotes a constant,

$$au = \int a\frac{du}{dx}dx.$$

But

$$u = \int \frac{du}{dx}dx,$$

and, accordingly,

$$(6) \qquad \int a\frac{du}{dx}\,dx = a\int\frac{du}{dx}dx.$$

This formula shows that any constant factor of a function given for integration may be written before or after the sign of integration.

EXAMPLES

1. $\int\left(a\frac{du}{dx} + b\frac{dv}{dx} - c\frac{du}{dx}\right)dx = a\int\frac{du}{dx}dx + b\int\frac{dv}{dx}dx - c\int\frac{du}{dx}dx.$

2. $\int(x + \sin x)dx = \int xdx + \int \sin xdx = \frac{x^2}{2} - \cos x + C.$

3. $\int(ax^2 + bx + c)dx = \int ax^2dx + \int bxdx + \int cdx$

$$= \frac{ax^3}{3} + b\frac{x^2}{2} + cx + C.$$

4. $\int\left(ax^2 + \frac{b}{x}\right)dx = \int ax^2dx + \int\frac{b}{x}dx = \frac{ax^3}{3} + b\log_e x + C.$

5. $\int(a\cos x + b\sin x)dx = a\sin x - b\cos x + C.$

EXERCISES XIX

NOTE. In these exercises, and in all integrations, it is always a sufficient proof of the correctness of a result, to show (by differentiation) that the derivative of the result obtained is the function given for integration.

Write the integrals of the following functions:

1. x^3.	3. $3x$.	5. ax^2.	7. $-cx$.
2. x^7.	4. x^{-5}.	6. $x^k - d$.	8. $3 + x$.

9. $x - 5$.

10. $x^2 - 12x + 9$.

11. $x^3 + 3x^2 + 4x - 6$.

12. $\frac{4}{x}$.

13. $\frac{x}{4}$.

14. $\frac{7 - 3x}{2x}$.

15. $\frac{3x^3 - 9x + 5}{4x}$.

Write the result of the operations indicated in the following expressions:

16. $\int 3 \cos x dx$.

17. $\int \left(x + \frac{1}{x}\right) dx$.

18. $\int \left(x^2 - \frac{1}{x^2}\right) dx$.

19. $\int \left(\frac{a}{\cos^2 x} + \frac{b}{\sin^2 x}\right) dx$.

20. $\int \frac{a^2}{1 + x^2} dx$.

21. $\int (7 \cos x - 4 \sin x + 1)\, dx$.

22. $\int \left(\frac{5}{x} + 7\right) dx$.

23. $\int \frac{3}{2\sqrt{1 - x^2}} dx$.

24. $\int ae^x\, dx$.

25. $\int 5(\sin x + \cos x)\, dx$.

26. $\int \frac{dx}{3}$.

27. $\int \frac{dx}{3x}$.

28. $\int (e^x + a^x + x^a)\, dx$.

ART. 2. **Integration by the introduction of new variables.** The determination of the integral can often be facilitated by the introduction of new variables, analogously to the process which has already been explained for the Differential Calculus (pp. 152–153).

Let there be given for integration

$$\int f(x)\, dx,$$

and suppose that the desired integral is $\phi(x)$, so that we have

(1) $$\phi(x) = \int f(x)\, dx,$$

(2) or $$f(x) = \frac{d}{dx}\phi(x).$$

Now put

(3) $$x = \psi(u).$$

Then $f(x)$ as well as $\phi(x)$ are functions of u, and we have (p. 152)

$$\frac{d\phi(x)}{du} = \frac{d\phi(x)}{dx} \cdot \frac{dx}{du};$$

or, making use of (2),

$$\frac{d\phi(x)}{du} = f(x)\frac{dx}{du}.$$

Whence, integrating with respect to u,

$$\phi(x) = \int f(x)\frac{dx}{du}\,du. \tag{4}$$

Equating values of $\phi(x)$ from (1) and (4), we have

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du, \tag{5}$$

where, of course, at some convenient point in the simplification of the expression under the integral sign in the right member, x must be replaced by its value in terms of u from (3), so that the function to be integrated with respect to u is finally expressed in terms of u alone.

We now apply this method to several examples.

I. Given $$\int (a + x)^n\,dx.$$

Put $$a + x = u,$$

$$\frac{dx}{du} = 1.$$

Hence $$\int (a + x)^n\,dx = \int u^n \cdot \frac{dx}{du}\,du = \int u^n\,du,$$

and $$\int u^n\,du = \frac{u^{n+1}}{n+1}.$$

Restoring the values of u,

(6) $$\int (a+x)^n dx = \frac{(a+x)^{n+1}}{n+1} + C.^*$$

In particular, we have for various values of n,

$$\int (a+x)\,dx = \frac{(a+x)^2}{2} + C,$$

$$\int (a+x)^2 dx = \frac{(a+x)^3}{3} + C,$$

$$\int \frac{dx}{(a+x)^2} = \int (a+x)^{-2} dx = -\frac{1}{a+x} + C.$$

$$\int \frac{dx}{(a+x)^3} = \int (a+x)^{-3} dx = -\frac{1}{2(a+x)^2} + C,$$

etc., etc.

II. The integral

$$\int (a-x)^n dx$$

may be treated analogously.

We put $$a - x = u,$$

whence $$\frac{dx}{du} = -1,$$

$$\int (a-x)^n dx = \int u^n \cdot (-1) \cdot du = -\int u^n du,$$

or $$\int (a-x)^n dx = -\frac{u^{n+1}}{n+1};$$

and, replacing u by its value,

(7) $$\int (a-x)^n dx = -\frac{(a-x)^{n+1}}{n+1} + C,$$

* It is most convenient not to add the constant, until the final form of the result is reached.

and in particular,

$$\int (a-x)dx = -\frac{(a-x)^2}{2} + C,$$

$$\int (a-x)^2 dx = -\frac{(a-x)^3}{3} + C,$$

$$\int \frac{dx}{(a-x)^2} = \int (a-x)^{-2} dx = \frac{1}{a-x} + C,$$

$$\int \frac{dx}{(a-x)^3} = \int (a-x)^{-3} dx = \frac{1}{2(a-x)^2} + C,$$

etc., etc.

If $n = -1$, the integrals in both these cases lead to logarithms (cf. p. 176).

For example, if in the integral

$$\int \frac{dx}{a+x}$$

we put $$a + x = u,$$

and therefore, $$\frac{dx}{du} = 1,$$

we have

(8) $$\int \frac{dx}{a+x} = \int \frac{du}{u} = \log u = \log(a+x) + C.$$

III. To determine the integral

$$\int \frac{A dx}{a-x}$$

we put $$a - x = u.$$

Therefore, $$\frac{dx}{du} = -1,$$

and $$\int \frac{A dx}{a-x} = -A \int \frac{du}{u} = -A \log u = A \log \frac{1}{u},$$

(9) $$= A \log \frac{1}{a-x} + C.$$

We have already met the last integral in the consideration of the inversion of sugar (p. 182). There, however, we simply verified the result which we have here deduced.

IV. Given $$\int \tan x \, dx.$$

We write, in the first place,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx,$$

and put $$\cos x = u, \text{ or } x = \text{arc} \cos u,$$

whence

$$\frac{dx}{du} = -\frac{1}{\sqrt{1-u^2}} = -\frac{1}{\sqrt{1-\cos^2 x}} = -\frac{1}{\sin x}.*$$

We have then $$\int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\log u.$$

Consequently,†

(10) $$\int \tan x \, dx = \log \frac{1}{\cos x} = -\log \cos x + C.$$

V. Similarly, in $$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx,$$

we put $$\sin x = u,$$

whence $$\frac{dx}{du} = \frac{1}{\cos x},$$

and find

(11) $$\int \cot x \, dx = \log \sin x + C.$$

---

* This can be found also thus:

$$\frac{du}{dx} = -\sin x, \text{ and (p. 142) } \frac{dx}{du} = \frac{1}{\frac{du}{dx}} = -\frac{1}{\sin x} = -\frac{1}{\sqrt{1-u^2}}.$$

† Formula 29, Appendix.

VI. Given $$\int \sin x \cos x \, dx.$$

Put $\sin x = u$, whence $\dfrac{dx}{du} = \dfrac{1}{\cos x}$, and therefore,

$$\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} = \frac{\sin^2 x}{2} + C. \tag{12}$$

EXERCISES XX

Determine the following integrals:

1. $\int e^{2x} dx.$
2. $\int \dfrac{e^x + 1}{e^x + x} dx.$
3. $\int \dfrac{x^4 dx}{(x^5 - 73)^{73}}.$
4. $\int (x^2 - 5x)(2x - 5) dx.$
5. $\int \dfrac{2x}{\sqrt{a^2 - x^2}} dx.$
6. $\int (3x^2 + 5x - 1)^3 (6x + 5) dx.$
7. $\int (2x^3 - 7x^2 + 3)^n (6x^2 - 14x) dx.$
8. $\int \dfrac{dx}{\sqrt{a^2 + x^2}}$ (put $u - x = \sqrt{a^2 + x^2}$).
9. $\int \dfrac{\cos \sqrt{x}}{\sqrt{x}} dx$ (put $x = u^2$).
10. $\int x^2 \cos x^3 dx$ (put $x = u^{\frac{1}{3}}$).
11. $\int e^x \cos e^x$ (put $e^x = u$).
12. $\int n x^{n-1} \cos x^n$ (put $x^n = u$).
13. $\int \dfrac{\sin (\log x) dx}{x}$ (put $\log x = u$).
14. $\int \dfrac{1 + \cos x}{x + \sin x} dx.$
15. $\int \dfrac{x}{(1 - x)^4}$ (put $1 - x = u$).
16. $\int \dfrac{dx}{\sin mx}.$
17. $\int \sin^3 x \cos^2 x \, dx$ (put $\cos x = u$).
18. $\int \sqrt{a^2 - x^2} \, dx$ (put $x = a \sin u$).

ART. 3. **Integration by parts.** By integrating the formula for the differentiation of a product, *viz.* (p. 130)

$$\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}, \tag{1}$$

we find $$uv = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx,$$

the inverse of the analogous formula of the Differential Calculus. By writing this formula in the form

$$(2) \qquad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx,$$

we see at once that it expresses one integral in terms of another. If we know the integral in the right member, we can by its aid calculate the integral of the left member also. The method of applying this formula is to determine functions u and v such that the product $u\frac{dv}{dx}$ is equal to the function given for integration, and therefore an equation of the type (2) can be set up having our desired integral as its left member. How this is to be done in practice will appear best from the consideration of some examples.

I. In order to apply our formula to the integral

$$\int \log x\,dx,$$

we put
$$u = \log x,\ \frac{dv}{dx} = 1,$$

and then find
$$\frac{du}{dx} = \frac{1}{x}, \qquad v = x,$$

whence
$$\int \log x\,dx = x\log x - \int x\cdot\frac{1}{x}\,dx$$

$$(3) \qquad = x\log x - x.$$

II. As a second example, we take the integral

$$\int xe^x\,dx.$$

In this case we put

$$u = x, \qquad \frac{dv}{dx} = e^x,$$

and find
$$\frac{du}{dx} = 1,\ v = \int e^x\,dx = e^x.$$

Therefore it follows that (substituting in (2)),

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$(4) \qquad = xe^x - e^x.$$

No general rule can be given as to the manner in which an integral is to be divided into the two parts u and $\frac{dv}{dx}$ in order that the method may actually accomplish its object; this can be ascertained only by trial. It is sometimes advantageous to take unity as one of the factors; for example, $\int \log x\,dx$ was found thus. It is, however, clear that the function v must in every case be so chosen that it is possible to determine it from its derivative $\frac{dv}{dx}$, and that the integral to which the required integral is reduced must be known or, at least, must be more easily determinable than the required one.

III. If, for example, we put in the last integral,

$$u = e^x, \quad \frac{dv}{dx} = x,$$

which is perfectly permissible, then v is in this case also easily determined, for

$$v = \int x\,dx = \frac{x^2}{2}, \text{ and further, } \frac{du}{dx} = e^x,$$

whence $$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx.$$

But in this way we have referred the required integral back to one that is evidently more complex; such a substitution has therefore no practical value for the determination of the integral sought.

The method of determining integrals just described is called the **method of integration by parts.**

IV. We next employ this method in determining the integral,

$$\int x \sin x \, dx.$$

By putting $u = x,\ \dfrac{dv}{dx} = \sin x,$

we find $\dfrac{du}{dx} = 1,\ v = \int \sin x \, dx = -\cos x,$

and therefore

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$

(5) $$= -x \cos x + \sin x.$$

V. The integral $\int x^2 \sin x \, dx$

can be treated in a similar manner. We put

$$u = x^2,\ \frac{dv}{dx} = \sin x.$$

Then, $\dfrac{du}{dx} = 2x,\ v = \int \sin x \, dx = -\cos x;$

whence $\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$

In order to determine the integral in the right-hand member, we put

$$u = 2x,\ \frac{dv}{dx} = \cos x;$$

whence $\dfrac{du}{dx} = 2,\qquad v = \int \cos x \, dx = \sin x,$

and find accordingly,

$$\int 2\,x \cos x\,dx = 2\,x \sin x - \int 2 \sin x\,dx$$
$$= 2\,x \sin x + 2 \cos x;$$

obtaining as the final result

$$(6) \qquad \int x^2 \sin x\,dx = -\,x^2 \cos x + 2\,x \sin x + 2 \cos x.$$

EXERCISES XXI

Integrate by parts:

1. $\int x \cos nx\,dx.$
2. $\int x^2 \sin nx\,dx.$
3. $\int x^2 e^x\,dx.$
4. $\int x^3 \log x\,dx.$
5. $\int \operatorname{arc} \sin x\,dx.$
6. $\int x^3 e^{ax}\,dx.$
7. $\int x^2 \operatorname{arc} \sin x\,dx.$

Show that:

8. $\int xe^{nx}dx = \dfrac{e^{nx}(nx-1)}{n^2}.$
9. $\int x^n \log x\,dx = \dfrac{x^{n+1}[(n+1)\log x - 1]}{(n+1)^2}.$
10. $\int \operatorname{arc} \cot x\,dx = x \operatorname{arc} \cot x + \frac{1}{2} \log (1 + x^2).$

ART. 4. **On special artifices.** The examples hitherto treated are already sufficient to show that the evaluation of integrals is markedly more complicated than the formation of derivatives. This corresponds to the character of the Integral Calculus as dealing with an inverse problem. In particular, we do not have in the Integral Calculus a definite method corresponding to that for forming derivatives, enabling us to form the integrals of arbitrary functions.

Consequently, the determination of integrals makes drafts on our resources quite different from those made in the formation of derivatives. Mathematical science is as yet far from able to determine the integrals of arbitrarily assigned functions, and it does not even fall within the scope of this work to treat or to enumerate all the results which have thus far been obtained for integrals of special form, more or less complicated. We add, however, several simple examples illustrating some of the artifices by which many integrals may be determined.

ART. 5. **Integration by transformation of the function to be integrated.** It frequently happens that the function to be integrated may be transformed so as to bring the integral under results previously found.

I. To determine

(1) $$\int \frac{dx}{\sin x \cos x}$$

We have

$$\int \frac{dx}{\sin x \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx\,^*$$

(2) $$= \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{\sin x} dx$$

and accordingly, introducing the results of p. 190,

$$\int \frac{dx}{\sin x \cos x} = \log \sin x - \log \cos x = \log \frac{\sin x}{\cos x}$$

(3) $$= \log \tan x.$$

II. This result will enable us to compute the integrals,

(4) $$\int \frac{dx}{\sin x} \text{ and } \int \frac{dx}{\cos x}.$$

---

* Formula 28, Appendix.

We have

(5) $$\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}.^{*}$$

Put $\frac{x}{2}=u$, and hence $\frac{dx}{du}=2$.

(6) $$\int\frac{dx}{\sin x}=\int\frac{du}{\sin u\cos u}=\log\tan u+C=\log\tan\frac{x}{2}+C.$$

In order to determine $\int\frac{dx}{\cos x}$,
we first have †

(7) $$\int\frac{dx}{\cos x}=\int\frac{dx}{\sin\left(\frac{\pi}{2}+x\right)}.$$

Now we put
$$\frac{\pi}{2}+x=u,\ \text{whence}\ \frac{dx}{du}=1;$$
and, therefore,

(8) $$\int\frac{dx}{\cos x}=\int\frac{du}{\sin u}=\log\tan\frac{u}{2}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

III. To find $\int\sqrt{a^2-x^2}\,dx$.

Integrating by parts,

(9) $$\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx.$$

(10) $$\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx=\int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$
$$=\int\frac{a^2}{\sqrt{a^2-x^2}}\,dx-\int\sqrt{a^2-x^2}\,dx$$
$$=a^2\arcsin\frac{x}{a}-\int\sqrt{a^2-x^2}\,dx.$$

* Formula 38, Appendix. † Formula 17, Appendix.

Substituting in (9),

$$(11)\quad \int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \text{ arc sin } \frac{x}{a} - \int \sqrt{a^2 - x^2}\, dx.$$

Whence

$$(12)\qquad \int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \text{ arc sin } \frac{x}{a}.$$

IV. We could find $\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx$, by subtracting (12) from (9) above. We may also find it as follows:

Integrating by parts,

$$\begin{aligned} \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx &= -x\sqrt{a^2 - x^2} + \int \sqrt{a^2 - x^2}\, dx \\ &= -x\sqrt{a^2 - x^2} + \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\, dx \\ &= -x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx \\ &= -x\sqrt{a^2 - x^2} + a^2 \text{ arc sin } \frac{x}{a} - \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx. \end{aligned}$$

Whence

$$(13)\qquad \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \text{ arc sin } \frac{x}{a}.$$

V. To find $\int \sqrt{a^2 + x^2}\, dx$.

Integrating by parts,

$$(14)\qquad \int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}}\, dx.$$

We have also,

$$\int \sqrt{a^2 + x^2}\, dx = \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}}\, dx = a^2 \int \frac{dx}{\sqrt{a^2 + x^2}} + \int \frac{x^2}{\sqrt{a^2 + x^2}}\, dx.$$

Adding, and dividing by 2,

$$(15)\quad \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}}$$
$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right).$$

(See Ex. 8, p. 191.)

ART. 6. **Formulæ of reduction.** In many cases, the given integral may be expressed in terms of known functions and a new integral. The determination of the given integral is thus reduced to the determination of the new integral, and the formula connecting the two is called a **formula of reduction.** The method of integration by parts is an illustration.

I. Integration by parts will also lead to many special formulæ of reduction. Thus, first integrating by parts, and then in the resulting integral, multiplying numerator and denominator by $\sqrt{a^2-x^2}$, and simplifying, we find

$$(1)\quad \int\frac{x^n}{\sqrt{a^2-x^2}}\,dx = -\frac{x^{n-1}}{n}\sqrt{a^2-x^2} + \frac{(n-1)a^2}{n}\int\frac{x^{n-2}dx}{\sqrt{a^2-x^2}}.^*$$

By applying this formula repeatedly, we should at last come either to $\int\frac{dx}{\sqrt{a^2-x^2}}$ or to $\int\frac{x}{\sqrt{a^2-x^2}}\,dx$ (according as n is even or odd), both of which are known.

II. We have

$$(2)\quad \frac{d}{dx}\tan x\sec^{n-2}x = (n-2)\tan^2 x\sec^{n-2}x + \sec^n x$$
$$= (n-1)\sec^n x - (n-2)\sec^{n-2}x;$$

---

* Equation (1) does not hold for $n=0$, since for this value of n, the coefficients become infinite. It is usually possible to see readily for what value or values of n, if any, the similar formulæ which we shall have, do not hold.

whence, by integrating,

(3) $$\tan x \sec^{n-2} x = n - 1\int \sec^n x\, dx - (n-2)\int \sec^{n-2} x\, dx,$$

or, $$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$$

By repeated application of this formula of reduction, the determination of the integral will be reduced either to that of $\int dx$ or of $\int \sec x\, dx$, both of which are known.

There are many varieties of formulæ of reduction, but the scope of our work does not permit us to take up even the simpler ones of them.

ART. 7. **Integration by inspection.** I. If the function to be integrated can be separated by inspection into two factors, one of which is the derivative of the other, then the integral is equal to one-half the square of the latter factor. In symbols,

$$\int u \frac{du}{dx}\, dx = \frac{u^2}{2},$$

and more generally, $$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1},$$

as may readily be proved by differentiating the right members.

EXAMPLES

1. $\int (x^3 + 2x)(3x^2 + 2)\, dx = \frac{(x^3 + 2x)^2}{2}.$

2. $\int \sin x \cos x\, dx = \frac{\sin^2 x}{2}.$

3. $\int \frac{\sin x}{\cos^3 x}\, dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x}\, dx = \int \tan x \sec^2 x\, dx = \frac{\tan^2 x}{2}.$

4. $\int x(a^2 + x^2)^{\frac{5}{2}}\, dx = \frac{(a^2 + x^2)^{\frac{7}{2}}}{7}.$

II. If the function to be integrated can be written as a fraction whose numerator is the derivative of the denominator, then the integral is the logarithm of the denominator. In symbols,

$$\int \frac{\dfrac{du}{dx}}{u}\,dx = \log u.$$

EXAMPLES

1. $\displaystyle\int \frac{e^x}{e^x + 5} = \log(e^x + 5).$ 2. $\displaystyle\int \frac{3x^2 - 7}{x^3 - 7x}\,dx = \log(x^3 - 7x).$

The various preceding calculations can be somewhat abbreviated in form by not explicitly introducing the new functional symbol u. This is analogous to the abbreviation spoken of in the Differential Calculus (p. 154), and the cautions there given to the student with regard to using this method may be repeated with double emphasis here. In addition to making the mental transformation of the function into terms of u, the value of $\dfrac{dx}{du}$ must be mentally computed and substituted, making a process of several steps in which mistakes may easily occur. The student should not attempt to use this method, even in simple cases, until he is quite skilful in determining the integral by the formal use of the function u.

EXERCISES XXII

Show that

1. $\displaystyle\int \frac{x^n}{\sqrt{a^2 + x^2}}\,dx = \frac{x^{n-1}}{n}\sqrt{a^2 + x^2} - \frac{(n-1)a^2}{n}\int \frac{x^{n-2}}{\sqrt{a^2 + x^2}}\,dx.$

2. $\displaystyle\int x^n\sqrt{a^2 - x^2}\,dx = \frac{x^{n+1}}{n+2}\sqrt{a^2 + x^2} + \frac{a^2}{n+2}\int \frac{x^n dx}{\sqrt{a^2 - x^2}}.$

HINT: Integrate by parts, then use the formula of reduction (1), p. 199.

3. $\displaystyle\int x^n\sqrt{a^2 + x^2}\,dx = \frac{x^{n+1}}{n+2}\sqrt{a^2 + x^2} + \frac{a^2}{n+2}\int \frac{x^n dx}{\sqrt{a^2 + x^2}}.$

Find the value of:

4. $\int \frac{x^4}{\sqrt{a^2 - x^2}}\,dx.$

5. $\int \frac{x^3}{\sqrt{1 - x^2}}\,dx.$

6. $\int \sec^4 x\,dx.$

7. $\int x^3\sqrt{4 - x^2}\,dx.$

8. $\int x\sqrt{a^2 + x^2}\,dx.$

9. $\int \frac{1}{3 - 5x}\,dx.$

10. $\int \frac{5}{2x - 4}\,dx.$

11. $\int \left(\frac{1}{x + 2} + \frac{1}{x - 2}\right)dx.$

12. $\int \frac{\log x}{x}\,dx.$

13. $\int e^x(e^x + a)\,dx.$

14. $\int \left(\frac{1}{a + bx} - \frac{1}{a - bx}\right)dx.$

15. $\int \frac{\text{arc}\cos x}{\sqrt{1 - x^2}}\,dx.$

16. $\int \frac{(2ax - a)\,dx}{(ax^2 - ax + 1)}.$

17. $\int \frac{x^4 dx}{x^5 - 73}.$

18. $\int \frac{x\,dx}{x^2 - a^2}.$

19. $\int \frac{3x^3}{\sqrt{9 + x^2}}\,dx.$

20. $\int \sec^5 x\,dx.$

21. $\int \sec^3 6x\,dx.$

22. $\int \frac{3x^2 dx}{a^3 + x^3}.$

23. $\int \frac{x}{\sqrt{1 + x^2}} \log \sqrt{1 + x^2}.$

24. $\int \cos^2 x \sin x\,dx.$

25. $\int \sin^2 x \cos x\,dx.$

26. $\int \cos^n x \sin x\,dx.$

27. $\int \frac{\text{arc}\tan x}{1 + x^2}\,dx.$

28. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$

29. $\int \frac{1 - \cos x}{x - \sin x}\,dx.$

30. $\int \frac{(\log x)^n}{x}\,dx.$

31. $\int (x^2 + 3x - 1)^{\frac{2}{3}}(2x + 3)\,dx.$

32. $\int x^{-3}(x^{-2} + 5) - \tfrac{11}{6}\,dx.$

Show that:

33. $\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$

34. $\int \frac{x^2}{\sqrt{x^2 + a^2}}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right).$

35. $\int \cos^2 x\,dx = \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x.$

36. $\int \sin^2 dx = -\tfrac{1}{4}\sin 2x + \tfrac{1}{2}x.$

ART. 8. **Decomposition into partial fractions.** The integration of rational fractions is usually accomplished by breaking up the given fraction into a sum of simpler fractions. The following examples will make the method clear:

I. $$\int \frac{dx}{(a-x)(b-x)}, \quad (b \neq a).$$

We shall show first that numbers p and q can be found such that

(1) $$\frac{1}{(a-x)(b-x)} = \frac{p}{a-x} + \frac{q}{b-x}.$$

We notice that

$$\frac{p}{a-x} + \frac{q}{b-x} = \frac{bp + aq - (p+q)x}{(a-x)(b-x)}.$$

If p and q be chosen so that

(2) $$bp + aq = 1, \quad p + q = 0,$$

the numerator of the fraction last written has the value 1, and the equation (1) is established.

The desired values of p and q can always be found by solving the system (2) for the two unknown quantities p and q, with the results

(3) $$p = \frac{1}{b-a}, \quad q = -\frac{1}{b-a}.$$

We find, therefore, that

(4) $$\begin{aligned}\int \frac{dx}{(a-x)(b-x)} &= \int \frac{1}{b-a} \cdot \frac{dx}{a-x} - \int \frac{1}{b-a} \cdot \frac{dx}{b-x} \\ &= -\frac{1}{b-a}\log(a-x) + \frac{1}{b-a}\log(b-x) \\ &= \frac{1}{b-a}\log\frac{b-x}{a-x} + C.\end{aligned}$$

II. In a similar manner, the integral

$$\int \frac{A + Bx}{(a - x)(b - x)} dx, \qquad (b \neq a),$$

where A and B are given constants, may be determined. Here, also, in the first place we seek to determine two numbers p and q such that

$$\frac{A + Bx}{(a - x)(b - x)} = \frac{p}{a - x} + \frac{q}{b - x}. \tag{5}$$

Just as above, we find that if we can determine p and q to satisfy the conditions

$$pb + qa = A, \quad -(p + q) = B, \tag{6}$$

the relation (5) will be established. By solving the equations (6), regarding p and q as unknown quantities, we find

$$p = \frac{A + Ba}{b - a}, \quad q = \frac{A + Bb}{a - b}. \tag{7}$$

For brevity, we shall still retain the symbols p and q in our work, they having now the values just found.

We have then

$$\int \frac{A + Bx}{(a - x)(b - x)} dx = \int \frac{p\,dx}{a - x} + \int \frac{q\,dx}{b - x} \tag{8}$$

$$= p \log \frac{1}{a - x} + q \log \frac{1}{b - x} + C,$$

where p and q have the values found at (7).

The resolution of the fraction under the integral sign into the sum of two fractions, whose denominators are respectively the two factors of the denominator of the given fraction is known as **Decomposition into Partial Fractions.***

* This problem is the inverse of the problem to reduce given fractions to a common denominator and add.

The following are special cases of the above results:

$$\text{(9)} \qquad \int \frac{dx}{(1-x)(2-x)} = \log \frac{2-x}{1-x} + C,$$

$$\int \frac{1-2x}{(1-x)(2-x)} dx = \log(1-x) + 3\log\frac{1}{2-x} + C$$

$$\text{(10)} \qquad = \log\frac{1-x}{(2-x)^3} + C.^*$$

III. The method explained above can be extended at once to the case that the denominator of the fraction under the sign of integration can be broken up into more than two factors. We shall treat the case of three factors, from which the method to be followed in the case of more than three factors is clear.

We consider the integral

$$\int \frac{A + Bx + Cx^2}{(a-x)(b-x)(c-x)}\, ax,$$

where A, B, C, a, b, c, are given numbers (a, b, c unequal).

We seek to determine three numbers, p, q, r, such that

$$\text{(11)} \qquad \frac{A + Bx + Cx^2}{(a-x)(b-x)(c-x)} = \frac{p}{a-x} + \frac{q}{b-x} + \frac{r}{c-x}$$

shall be true for all values of x. This could be done analogously to the previous examples by reducing the right member to a common denominator, and then equating the coefficient of x^2 in the resulting numerator with C, that of x with B, and the term free from x with A. Three equations of the first degree would result, from which the values of

---

* Formulæ 5 and 7, Appendix.

the three unknown quantities, p, q, r, could be determined. But they may be determined also by the following method:

Multiplying the equations (9) by $a-x$, we obtain

$$\text{(12)} \qquad \frac{A+Bx+Cx^2}{(b-x)(c-x)}=p+q\frac{a-x}{b-x}+r\frac{a-x}{b-x}.$$

Since relation (11) is to hold for all values of x, the relation just written must do so too, and we have, for $x=a$ in particular

$$\frac{A+Ba+Ca^2}{(b-a)(c-a)}=p.$$

Quite similarly we find

$$\text{(13)} \qquad q=\frac{A+Bb+Cb^2}{(a-b)(c-b)}, \quad r=\frac{A+Bc+Cc^2}{(a-c)(b-c)}.$$

For brevity, we still retain the symbols p, q, r, to denote these values, and have

$$\text{(14)} \qquad \int\frac{A+Bx+Cx^2,}{(a-x)(b-x)(c-x)}dx$$

$$=\int\frac{p\,dx}{a-x}+\int\frac{q\,dx}{b-x}+\int\frac{r\,dx}{c-x}$$

$$=p\log\frac{1}{a-x}+q\log\frac{1}{b-x}+r\log\frac{1}{c-x}+C.$$

Example: To determine

$$\text{(15)} \qquad \int\frac{1-2x+3x^2}{(1-x)(2-x)(3-x)}dx.$$

Here $A=1$, $B=-2$, $C=3$, $a=1$, $b=2$, $c=3$.

Accordingly, $p=1$, $q=-9$, $r=11$;

and, consequently,

$$(16)\quad \int \frac{1-2x+3x^2}{(1-x)(2-x)(3-x)}dx$$

$$=\int \frac{dx}{1-x}-\int \frac{9\,dx}{2-x}+\int \frac{11\,dx}{3-x}$$

$$=\log\frac{1}{1-x}+9\log(2-x)+11\log\frac{1}{3-x}+C$$

$$=\log\frac{(2-x)^9}{(1-x)(3-x)^{11}}+C.*$$

IV. We take up next the case in which two of the factors of the denominator are equal.

Let there be given for integration

$$(17)\qquad \int \frac{dx}{(a-x)^2(b-x)},\qquad (b \neq a).$$

We put

$$(18)\qquad \frac{1}{(a-x)^2(b-x)}=\frac{p}{(a-x)^2}+\frac{q}{a-x}+\frac{r}{b-x},$$

where p, q, r, are numbers to be determined; from this by clearing of fractions,

$$(19)\qquad 1=p(b-x)+q(a-x)(b-x)+r(a-x)^2.$$

This equation is to hold for all values of x. For $x=a$, in particular, we have

$$1=p(b-a)\quad\text{or}\quad p=\frac{1}{b-a},$$

and for $x=b$, $\quad 1=r(a-b)^2\quad$ or $\quad r=\dfrac{1}{(a-b)^2},$

and for $x=0$, $\quad 1=pb+q\,ab+ra^2,$

* Formulæ 5 and 7, Appendix.

whence, by substituting the values found for p and r, we find that

$$q = -\frac{1}{(a-b)^2}.$$

Therefore

$$\frac{1}{(a-x)^2(b-x)}$$

$$= \frac{1}{b-a}\cdot\frac{1}{(a-x)^2} - \frac{1}{(b-a)^2}\,\frac{1}{(a-x)} + \frac{1}{(b-a)^2}\cdot\frac{1}{(b-x)}$$

and

$$(20)\quad \int\frac{dx}{(a-x)^2(b-x)}$$

$$= \frac{1}{(b-a)}\int\frac{dx}{(a-x)^2} - \frac{1}{(b-a)^2}\int\frac{dx}{a-x} + \frac{1}{(b-a)^2}\int\frac{dx}{b-x}$$

$$= \frac{1}{b-a}\cdot\frac{1}{a-x} - \frac{1}{(b-a)^2}\log\frac{1}{a-x} + \frac{1}{(b-a)^2}\log\frac{1}{b-x}$$

$$= \frac{1}{b-a}\cdot\frac{1}{a-x} - \frac{1}{(b-a)^2}\log\frac{b-x}{a-x} + C.$$

V. We treat next the integral

$$(21)\qquad \int\frac{A+Bx}{(a-x)^2(b-x)}dx, \quad (b \neq a).$$

Here, likewise, we put

$$(22)\qquad \frac{A+Bx}{(a-x)^2(b-x)} = \frac{p}{(a-x)^2} + \frac{q}{a-x} + \frac{r}{b-x},$$

and have, by clearing of fractions,

$$(23)\quad A + Bx = p(b-x) + q(a-x)(b-x) + r(a-x)^2;$$

whence, putting x equal to a, b, and zero in turn, we find that

$$p = \frac{A+Ba}{b-a}, \quad r = \frac{A+Bb}{(a-b)^2}, \quad q = -\frac{A+Bb}{(a-b)^2}.$$

Accordingly,

$$(24)\qquad \int \frac{A+Bx}{(a-x)^2(b-x)}\,dx$$

$$=\int \frac{A+Ba}{b-a}\cdot\frac{dx}{(a-x)^2}-\int\frac{A+Bb}{(a-b)^2}\cdot\frac{dx}{a-x}+\int\frac{A+Bb}{(a-b)^2}\cdot\frac{dx}{b-x}$$

$$=\frac{A+Ba}{b-a}\cdot\frac{1}{a-x}-\frac{A+Bb}{(a-b)^2}\log\frac{1}{a-x}+\frac{A+Bb}{(a-b)^2}\log\frac{1}{b-x}$$

$$=\frac{A+Ba}{b-a}\,\frac{1}{a-x}+\frac{A+Bb}{(a-b)^2}\log\frac{a-x}{b-x}+C.$$

VI. $$\int\frac{A+Bx+Cx^2}{(a+bx)^3}\,dx.$$

Put

$$(25)\qquad \frac{A+Bx+Cx^2}{(a+bx)^3}=\frac{p}{(a+bx)^3}+\frac{q}{(a+bx)^2}+\frac{r}{(a+bx)}.$$

Clearing of fractions,

$$(26)\qquad A+Bx+Cx^2=p+q(a+bx)+r(a+bx)^2.$$

Put $x=-\frac{a}{b}$,

$$p=A-\frac{Ba}{b}+\frac{Ca^2}{b^2}.$$

Put $x=0$,

$$A=p+qa+ra^2.$$

Put $x=1$,

$$A+B+C=p+q(a+b)+r(a+b)^2.$$

Substituting the value of p found above in the last two equations and solving, we find the values of q and r, *viz.*,

$$q=\frac{B}{b}-\frac{2\,aC}{b^2},$$

$$r=\frac{C^2}{b^2}.$$

Accordingly,

$$(27)\quad \int \frac{A + Bx + Cx^2}{(a + bx)^3} = \int \frac{p\,dx}{(a + bx)^3} + \int \frac{q\,dx}{(a + bx)^2} + \int \frac{r\,dx}{a + bx}$$

$$= \frac{p}{2\,b(a + bx)^2} + \frac{q}{b(a + bx)} + \frac{r}{b} \log (a + bx) + C',$$

where p, q, r have the values found above, and C' is added as the arbitrary constant of integration, to avoid confusion with the C given in the integral.

VII. We have found the unknown quantities by substituting special values of x. We could also find them by making use of the principle of algebra, that if two expressions are equal for all values of x, the coefficients of the different powers of x in one expression are respectively equal to the coefficients of the same powers of x in the other expression.

We determine by this method the integral

$$(28)\qquad \int \frac{10\,x^3 - 13\,x^2 - 45\,x + 66}{(x - 1)^3 (2\,x + 7)}.$$

Put

$$(29)\qquad \frac{10\,x^3 - 13\,x^2 - 45\,x + 66}{(x - 1)^3 (2\,x + 7)}$$

$$= \frac{p}{(x - 1)^3} + \frac{q}{(x - 1)^2} + \frac{r}{(x - 1)} + \frac{s}{(2\,x + 7)}.$$

Clearing of fractions, multiplying out the right member, and collecting,

$$(30)\quad 10\,x^3 - 13\,x^2 - 45\,x + 66$$

$$= (2\,r + s)x^3 + (2\,q + 3\,r - 3\,s)x^2$$

$$+ (2\,p + 5\,q - 12\,r + 3\,s)x + (7\,p - 7\,q + 7\,r - s).$$

Equating coefficients of like powers of x,

$$
(31)\qquad
\begin{aligned}
10 &= 2r + s,\\
-13 &= 2q + 3r - 3s,\\
-45 &= 2p + 5q - 12r + 3s,\\
66 &= 7p - 7q + 7r - s.
\end{aligned}
$$

Solving by the methods of elementary algebra, we find

$$p = 2,\ q = -5,\ r = 3,\ s = 4.$$

These values could have been found more readily by combining the two methods as follows:

Clearing the equation (29) of fractions, we have

$$
(32)\quad 10x^3 - 13x^2 - 45x + 66 = p(2x+7) + q(2x+7)(x-1) + r(2x+7)(x-1)^2 + s(x-1)^3.
$$

In this put $x = 1$, and thus

$$18 = 9p, \text{ or } p = 2.$$

Similarly, putting $x = -\frac{7}{2}$,

$$-\tfrac{2916}{8} = -\tfrac{729}{8}s, \text{ or } s = 4.$$

Equating coefficients of x^3 and x^2 in (30), we have

$$
\begin{aligned}
10 &= 2r + s,\\
-13 &= 2q + 3r - 3s,
\end{aligned}
$$

whence $\qquad r = 3,\ q = -5.$

We have not made use of the coefficients of the first power of x, and of the term free from x. They give rise to the last two of equations (31), and must be satisfied by the values of p, q, r, s, which we have found. This affords a check against numerical mistakes in the calculation.

Returning now to our integral, we find

$$(33)\quad \int \frac{10x^3 - 13x^2 - 45x + 66}{(x-1)^3(2x+7)}\,dx$$

$$= -(x-1)^{-2} + 5(x-1)^{-1} + 3\log(x-1) + 2\log(2x+7) + C.$$

VIII. In all the examples which we have treated, the degree of the numerator of the fraction given for integration has been lower than that of the denominator. If this should not be the case, the fraction given can, by ordinary division, be expressed as the sum of a polynomial and a fraction whose numerator is of lower degree than its denominator, and the methods of this paragraph can be applied to the latter fraction. A single example will illustrate the process sufficiently,

$$(34)\qquad \int \frac{2x^4 - 6x^3 - x^2 + 18x - 3}{x^2 - 3x + 2}\,dx.$$

$$\frac{2x^4 - 6x^3 - x^2 + 18x - 3}{x^2 - 3x + 2} = 2x^2 - 5 + \frac{3x+7}{x^2 - 3x + 2}.$$

By the method of partial fractions, we find

$$(35)\qquad \frac{3x+7}{x^2 - 3x + 2} = \frac{13}{x-2} - \frac{10}{x-1}.$$

Accordingly,

$$(36)\quad \int \frac{2x^4 - 6x^3 - x^2 + 18x - 3}{x^2 - 3x + 2}$$

$$= \int (2x^2 - 5)\,dx + \int \frac{3x+7}{x^2 - 3x + 2}\,dx$$

$$= \tfrac{2}{3}x^3 - 5x + 13\log(x-2) - 10\log(x-1) + C.$$

The examples which we have solved will suffice to show the student how to apply this method in all the cases that

4. $\int \frac{dx}{-15 + 8x - x^2} = \int \frac{dx}{1 - (x-4)^2} = \log \sqrt{\frac{x-3}{5-x}}.$

5. $\int \frac{(x^2 - 2)dx}{(x^2 - 2x + 1)^2} = \frac{1 + 3x - 3x^2}{3(x-1)^3},$ (use partial fractions).

6. $\int \frac{dx}{x^2 + 4x + 2} = \frac{1}{2\sqrt{2}} \log \frac{x + 2 - \sqrt{2}}{x + 2 + \sqrt{2}}.$

7. $\int \frac{x\,dx}{1 + \sqrt{1+x}} = (1 + x)[\tfrac{2}{3}\sqrt{1+x} - 1].$

8. $\int \frac{\sin x}{\cos^2 x} dx = \sec x.$

9. $\int \frac{x\,dx}{(x-1)(x^2+4)} = \tfrac{1}{5} \log (x-1) - \tfrac{1}{10} \log (x^2+4) + \tfrac{2}{5} \arctan \frac{x}{2}.$

10. $\int \cos (px + q)dx = \frac{1}{p} \sin (px + q).$

11. $\int \sec^6 x\,dx = \tan x + \tfrac{2}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x.$

12. $\int (1 - \cos x)^2 dx = \frac{\sin 2x}{4} - 2 \sin x + \frac{3x}{2}.$

13. $\int \frac{x\,dx}{\sin^2 x} = -x \cot x + \log \sin x.$

14. $\int \frac{x\,dx}{\cos^2 x} = x \tan x + \log \cos x.$

15. $\int \frac{\cos x\,dx}{\sqrt{\sin x}} = 2\sqrt{\sin x}.$

16. $\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3}.$

17. $\int \frac{dx}{x \log x} = \log (\log x).$

18. $\int (\log x)^2 dx = x(\log x)^2 - 2x \log x + 2x.$

19. $\int x^2\sqrt{x+2}\,dx = \frac{2(x+2)^{\frac{3}{2}}}{105}(15x^2 - 24x + 32).$ Put $\sqrt{x+2} = u.$

20. $\int \frac{1 - x^{\frac{1}{2}}}{1 - x^{\frac{1}{3}}} dx = 6\left[\frac{x^{\frac{7}{6}}}{7} + \frac{x^{\frac{5}{6}}}{5} - \frac{x^{\frac{2}{3}}}{4} + \frac{x^{\frac{1}{2}}}{3} - \frac{x^{\frac{1}{3}}}{2} + x^{\frac{1}{6}} - \log (x^{\frac{1}{6}} + 1)\right].$

Put $x^{\frac{1}{6}} = u.$

CHAPTER VII

SOME APPLICATIONS OF THE INTEGRAL CALCULUS

ART. 1. **The attraction of a rod.** If two material points of mass m and m', respectively, are at a distance r from each other, Newton's Law of Gravitation tells us that an attraction A, whose amount is

$$A = \frac{mm'}{r^2}, \tag{1}$$

exists between them. With this fact given, we take up the problem:

To determine the amount of attraction exerted by a straight, homogeneous rod of uniform thickness and of length l upon a material point P of mass m, which is situated in the line of direction of the rod and at the distance a from its nearer end.

We take the length of the rod as the variable and denote it by x. Since the attraction depends upon the value of x, we let $F(x)$ denote the function that we wish to determine. We now ask ourselves by how much the attraction is increased when the length of the rod is augmented by h at its further end, the distance between the nearer end of the rod and the point remaining constant. When the length of the rod is increased by h the total attraction becomes $F(x+h)$, and the attraction of the added piece of length h is therefore $F(x+h)-F(x)$. Let us suppose now this added piece to be replaced by a material point of equal mass

situated where it will exert the same attraction; this point will be somewhere between the two ends of the added piece. If M denote the mass of the unit-length of the rod, and if ϵ denote the distance of the material point from that end of the rod which is further from the point P, we have, by Newton's Law, that the added piece exerts on P the attraction

$$\frac{mMh}{(a+x+\epsilon)^2}. \tag{2}$$

We have thus found two different expressions for the attraction of the added piece. Equating these, we have

$$F(x+h)-F(x)=\frac{mMh}{(a+x+\epsilon)^2},$$

or

$$\frac{F(x+h)-F(x)}{h}=\frac{mM}{(a+x+\epsilon)^2}.$$

This holds true for every length h. When h approaches the limit zero, the limit of the left member is the derivative of $F(x)$. In the right member ϵ also approaches zero since, by definition, ϵ is positive and less than h. Hence,

$$\frac{dF(x)}{dx}=\frac{mM}{(a+x)^2}. \tag{3}$$

Integrating, $$F(x)=\int\frac{mM}{(a+x)^2}dx.$$

This integral is very easily found, and we have, finally,

$$A=F(x)=-\frac{mM}{a+x}+C. \tag{4}$$

It is apparent that a rod of length $x=0$ exerts the attraction, $A=0$. If, in equation (4), x be put equal to zero,

$$0=-\frac{mM}{a}+C, \tag{5}$$

and by subtraction of (5) from (4),

$$(6) \qquad A = mM\left(\frac{1}{a} - \frac{1}{a+x}\right).$$

The attraction of the rod of length l is found by the substitution of l for x in the last equation; then

$$(7) \qquad A = mM\left(\frac{1}{a} - \frac{1}{a+l}\right).$$

If the rod be very long, the fraction $\frac{1}{a+l}$ becomes very small, so that it may be neglected * in comparison with $\frac{1}{a}$. The equation then assumes the simple form

$$(8) \qquad A = \frac{mM}{a}.$$

A rod, then, whose length is very great in comparison with the distance of one of its ends from a material point lying in the prolongation of its axis, exerts an *attraction which is practically independent of its length and inversely proportional to the distance of the point from its nearer end.*

ART. 2. **The hypsometric formula.** *What is the atmospheric pressure at the height H above the earth's surface?*

Let the pressure (in centimeters of mercury) and the density (referred to mercury) of the air at the earth's surface be denoted by B and S respectively. Atmospheric pressure being due to the weight of the air, a column of air H cms. high and of the *uniform* density S, exerts a pressure

* In mathematical considerations we seek exact results, and never neglect anything; in physical considerations, where the results are at best only approximately accurate, it is often permissible to neglect quantities small enough not to affect the result appreciably.

equal to HS cms. of mercury; and if the air's density were the same at all elevations, its pressure would diminish S cms. of mercury for every centimeter of elevation. But, in reality, the density of the air also diminishes with a decrease of pressure; and in accordance with Boyle's Law (p. 3) the density and the pressure of the air are directly proportional to each other.

Let the pressure at the height x above the surface of the earth be denoted by $F(x)$, and the density by s. Then at a higher elevation $x+h$, (the density at which may be denoted by s',) the pressure would be diminished by sh, if throughout the additional elevation, the density were constantly s. But this is too great as the density decreases from s as its largest value. On the other hand, it would be diminished by $s'h$ if the density were constantly s'; but this is too little, since s' is the least density in the additional elevation. The true value must therefore be intermediate, as $(s-\epsilon)h$, where $0<\epsilon<s-s'$. We accordingly have

$$(1) \qquad F(x+h)-F(x)=-(s-\epsilon)h,$$

the right member being affected with the minus sign, inasmuch as the pressure decreases with increased height.

According to Boyle's Law,

$$(2) \qquad s:S=F(x):B,$$

whence

$$s=S\frac{F(x)}{B}.$$

Combining this equation with (1), we have

$$F(x+h)-F(x)=-\left(\frac{SF(x)}{B}-\epsilon\right)h,$$

or

$$(3) \qquad \frac{F(x+h)-F(x)}{h}=-\left(\frac{SF(x)}{B}-\epsilon\right)=\epsilon-\frac{SF(x)}{B}.$$

Allowing h to approach the limit zero, and bearing in mind that when h approaches zero, s' approaches s, and hence ϵ approaches zero, we have

$$\frac{dF(x)}{dx} = -\frac{SF(x)}{B};$$

and putting, for brevity,

$$F(x) = y,$$

$$\frac{dy}{dx} = -\frac{Sy}{B}.$$

Therefore (p. 142),

$$(4) \qquad \frac{dx}{dy} = -\frac{B}{Sy},$$

and integrating with respect to y,

$$(5) \qquad x = \int -\frac{B}{Sy}\,dy,$$

$$= -\frac{B}{S}\log y + C.$$

Now when $x = 0$, that is, at the earth's surface, $y = B$; substituting in (5),

$$(6) \qquad 0 = -\frac{B}{S}\log B + C$$

and subtracting (6) from (5), we have

$$x = \frac{B}{S}\log\frac{B}{y}.$$

For the required atmospheric pressure y at the elevation $x = H$, we have

$$(7) \qquad \log\frac{B}{y} = \frac{S}{B}H,$$

or

$$y = Be^{-\frac{S}{B}H}.$$

Further, by transforming equation (7) we may calculate the height H above the earth's surface from the observed atmospheric pressure; thus,

$$H = \frac{B}{S} \log \frac{B}{y}; \tag{8}$$

this is the so-called **hypsometric formula.**

In the above discussion we have neglected the influence of temperature, moisture, and the latitude of the station; these factors naturally complicate the solution greatly. To take them into account for average conditions, the factor $\frac{B}{S}$ may be replaced by 1,820,000; this includes also the multiplier necessary to pass to common logarithms; the formula then becomes

$$H = 1{,}820{,}000 \log_{10} \frac{B}{y}. \tag{9}$$

This formula gives the height in cms. when the barometer is read in cms. To exemplify its use, let it be required to ascertain the elevation above sea level at which the barometer would stand at 1 cm. In this case, $B = 76$ and $y = 1$; then

$$H = 182 \cdot 10^4 \cdot \log 76 = 182 \cdot 10^4 \cdot 1.88 = 3{,}421{,}600 \text{ cms.}$$

EXERCISES XXV

1. A ball whose mass is 1 gram is placed at a distance of 10 cms. from a homogeneous rod of mass 1000 grams and in direct line with it. What is the attraction between the ball and the rod if the length of the latter is (i.) 10, (ii.) 1000, (iii.) 10^{10} cms., respectively?

2. At what height in kilometers (1 kilometer = 100,000 cms.) is the pressure of the atmosphere equivalent to that of a column of mercury 1 micron (1 micron = 0.0001 cm.) high? (Such a pressure of mercury is about that obtainable in a very good vacuum.)

3. What is the average reading of the barometer at a station 5 kilometers above sea level?

ART. **3. Newton's law of cooling.** *Given a body at the temperature θ_1; the temperature of its surroundings being lower and constantly equal to θ_0, it is required to find the law according to which the temperature of the body falls.*

We assume, with Newton, that the rapidity with which the body loses heat depends upon the nature of the body, and is proportional to the excess of its temperature over that of its environment.

Let $W(t)$ denote the total amount of heat given out by the body from the beginning of cooling up to the time t, and $W(t+h)$ the amount of heat given out up to the time $(t+h)$; then $W(t+h)-W(t)$ will represent the heat given out during the interval of time h.

Let the temperatures at the times t and $t+h$ be θ and θ', respectively. Then if the temperature remained the same as at the beginning of the interval h, during that interval an amount of heat equal to

$$(1) \qquad W_1(t) = k(\theta - \theta_0)h$$

would be given out, where k is a constant multiplier, (factor of proportionality,) dependent for its value upon the nature of the body. W_1 is, however, too large, since the temperature falls. Likewise, if the temperature remained constant as at the end of the interval, the heat given off would be

$$(2) \qquad W_2(t) = k(\theta' - \theta_0)h;$$

but this is too small since θ' is the lowest temperature in the interval. The true amount of heat being thus greater than the latter and smaller than the former, is given by the expression

$$(3) \qquad W(t) = k(\theta - \theta_0 - \epsilon)h,$$

where $0 < \epsilon < \theta - \theta'$.

We saw also that the heat given out is

$$W(t+h)-W(t).$$

Equating these two expressions, we have

$$W(t+h)-W(t)=k(\theta-\theta_0-\epsilon)h,$$

or
$$\frac{W(t+h)-W(t)}{h}=k(\theta-\theta_0-\epsilon).$$

The limit of this as h approaches zero is (since at the same time θ' approaches θ so that ϵ approaches zero)

$$\frac{dW(t)}{dt}=k(\theta-\theta_0). \tag{4}$$

This equation gives us the derivative, with respect to t, of the function which expresses the heat given out in terms of the time; but as the right member is not expressed in terms of t, we cannot find the function itself (*i.e.* integrate) directly. The following considerations, however, enable us to utilize this equation to attain the desired result; *viz.* to find the relation between the time which has elapsed and the temperature at that time.

The amount of heat $W(t)$ given out up to a certain time t may be regarded as a function of the temperature θ at that time as well as of the time t itself. In particular, the total amount of heat given off by the body in cooling down from the temperature θ (which may be taken as a variable) to the temperature of the environment θ_0 is *

$$W=mc(\theta_0-\theta), \tag{5}$$

* When it is more convenient (and can be done without confusion), a gain in simplicity of notation is often made by omitting to indicate the independent variable in the functional symbol. Thus, instead of writing $W(t)$, we may write simply W. If $t=\phi(\theta)$, we have $W(t)=W\{\phi(\theta)\}$, and for this also we may write simply W, provided no confusion arises through doing so; the context indicating whether W is to be regarded as a function of t or of θ.

where m denotes the mass of the body and c its specific heat.*

Differentiating this equation with respect to θ, we have

(6) $$\frac{dW}{d\theta} = -mc.$$

But θ itself, *i.e.* the temperature of the body at any instant, is a function of the time t which has elapsed since the cooling began. W is therefore a function of a function, and

$$\frac{dW}{dt}\frac{dt}{d\theta} = \frac{dW}{d\theta},$$

or $$\frac{dt}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{dW}{dt}}; \quad \frac{d\theta}{dt} = \frac{\frac{dW}{dt}}{\frac{dW}{d\theta}}.$$

Substituting from equations (3) and (6), we have

(7) $$\frac{dt}{d\theta} = -\frac{mc}{k(\theta - \theta_0)}; \quad \frac{d\theta}{dt} = -\frac{k}{mc}(\theta - \theta_0).$$

Before integrating, we remark that the mass m is entirely independent of the temperature while the specific heat c is very nearly so; regarding them both therefore as constants and integrating equation (7), we obtain

(8) $$t = -\frac{mc}{k}\int\frac{d\theta}{\theta - \theta_0},$$

or (9) $$t = -\frac{mc}{k}\log(\theta - \theta_0) + C.$$

* The **specific heat** of any substance is defined to be the ratio between the amount of heat given out by a certain mass of it in cooling off through one degree of temperature, to the amount of heat given off by an equal mass of water cooling through the same temperature interval. The variations of specific heat with the temperature may be neglected in this problem.

Now when $t = 0$, then $\theta = \theta_1$; on substituting these particular values in equation (9), we have

$$(10) \qquad \frac{mc}{k} \log (\theta_1 - \theta_0) = C,$$

and after subtracting (10) from (9), we get

$$(11) \qquad t = \frac{mc}{k} \log \frac{\theta_1 - \theta_0}{\theta - \theta_0}.$$

We have derived equation (11) by the aid of a hypothesis which, although it seems in itself highly probable, may yet be subject to question. We cannot, however, test it directly, since it is impossible to determine experimentally the variations of temperature in very short intervals of time. Yet we may ascertain with considerable accuracy whether or not the equation is in correspondence with experimental facts. Thus, Winkelmann* made a series of observations, tabulated below, on the cooling of a body when the temperature of the environment was constantly kept at 0° C. and the initial temperature of the body was 19°.90 C.; *i.e.* $\theta_0 = 0°.00$; $\theta_1 = 19°.90$.

θ	t	$\frac{1}{t} \log_{10} \frac{\theta_1 - \theta_0}{\theta - \theta_0}$
18°.9	3.45	0.006490
16°.9	10.85	0.006540
14°.9	19.30	0.006509
12°.9	28.80	0.006537
10°.9	40.10	0.006519
8°.9	53.75	0.006502
6°.9	70.95	0.006483

* Wiedemann's *Annalen der Physik*, Vol. 44, p. 195. 1891.

The data given in the third column are computed from the observed values of t and θ. From (11), by a little transformation, we have

$$(12) \qquad 0.4343\,\frac{k}{mc}=\frac{1}{t}\log_{10}\frac{\theta_1-\theta_0}{\theta-\theta_0}$$

(0.4343 is, approximately, the modulus of Briggean logarithms).

As the left member consists of constant quantities, the right member must likewise be constant, which is seen above to be the case; the slight and irregular variations which occur are to be attributed to errors in the observation of the various values of the time and temperature.

The foregoing discussion may be regarded as a typical example of the introductory considerations of pp. 166 and 167, a hypothesis concerning a natural phenomenon receives a mathematical formulation, and thus leads to the establishment of an expression containing derivatives (a "differential equation"). But in order to compare the requirements of the hypothesis thus formulated with actual facts, in other words, to test our hypothesis by observation and experiment, we have to deduce by integration an equation that is freed from derivatives and contains only finite quantities which are directly accessible to experiment and observation.

Successful setting up of the differential equation depends upon the acumen of the investigator; thereafter, its integration is entirely a matter of mathematical calculation.

Exercise. In another series of observations with the temperature of the environment at 0° C., Winkelmann obtained the following results, the initial temperature being 14°.86 C.:

θ = 14°.38	13°.42	12°.44	11°.45	10°.26	9°.97
t = 130	405	703	1026	1197	1570

Calculate the values of $\dfrac{1}{t}\log_{10}\dfrac{\theta_1-\theta_0}{\theta-\theta_0}$.

ART. 4. **Concerning the general method of all these applications.** The quantity ϵ which has been used in each of the preceding problems is quite an important auxiliary in making the mathematical formulation of the physical facts. We know two values giving the changes which *would* occur if the change proceeded uniformly (on two different hypotheses) throughout the interval h, which is without restrictions as to size (in particular, the interval h is by no means supposed to be small). We know that one of these values is too large and the other too small, and that the true value is obtained by increasing or diminishing one of the factors by a quantity which we call ϵ. We know that a quantity ϵ exists which, when introduced into the formula in the manner just indicated, gives the true value, that ϵ is positive and less than a certain quantity, and that ϵ approaches zero if h approaches zero. We are usually not able, however, to specify the value of ϵ exactly; and this is not necessary, since the ϵ no longer occurs in the equation which is deduced from that in h by equating the limits of both its members. It is sufficient to find that some value of ϵ exists such that for it the expression in question gives the exact change which takes place in the interval h, and second, that ϵ approaches zero when h does so.

The following presentation will illustrate graphically the relation of ϵ to the other quantities. Taking the data from the above problem (Newton's law of cooling), and denoting the relation between the temperature and the time by $\theta = f(t)$, we let the curve CC' represent the graph of $k(\theta - \theta_0)$. Let $OA = t$ and $AB = h$. Then $AC = k(\theta - \theta_0)$ and $BD = k(\theta' - \theta_0)$; rectangle $ACFB = k(\theta - \theta_0)h$, and rectangle $AEDB = k(\theta' - \theta_0)h$. The quantity which we seek, denoted above by W, has been proved to be larger than one of these rectangles and smaller than the other. (It will be shown in the next chapter that W is precisely the area $ACGDB$.) If we move the line ED up parallel to itself, there must be some position, call it MH, such that rectangle $AMHB$ is exactly equal to W. For as the line ED moves up parallel to itself, the rectangle $EDBA$ increases continually, and passes from being smaller than W (at ED) to being larger than W (at CF). In doing so it must pass through a position in which the rectangle is just equal to W. The distance ME is the graphic equivalent of $k\epsilon$.

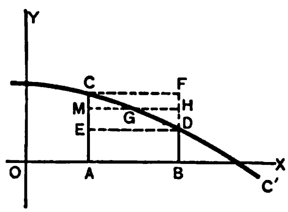

FIG. 47.

We may also show graphically that ϵ is less than $\theta - \theta'$. For, from the values of AC and BD above, we have $CE = k(\theta - \theta')$. But $ME = k\epsilon$, and $ME < CE$; consequently, $k\epsilon < k(\theta - \theta')$, or $\epsilon < \theta - \theta'$.

ART. 5. **Work done in the expansion of a perfect gas at a constant temperature.** When a gas kept under constant pressure p (the temperature also being constant) expands by the volume v, it is known that the work done is equal to the product pv; but if the temperature alone is kept constant, the pressure of a confined mass of gas will change continually as the volume passes from the value v_1 to the value v_2.

In accordance with our usual method of procedure, we assume the pressure to be constant during an expansion equal to h; the work then done may be put equal to ph. If the amount of work done in the expansion of the gas from the initial volume v_1 to the variable volume v be denoted by $L(v)$, and if p denote the pressure when the volume is v, and p' that when the volume is $(v+h)$, then putting $0<\epsilon<p-p'$, we have

$$L(v+h)-L(v)=(p-\epsilon)h$$

or

$$\frac{L(v+h)-L(v)}{h}=p-\epsilon.$$

As h approaches zero, this approaches the limit

(1) $$\frac{dL}{dv}=p,$$

and by integration,

(2) $$L=\int p\,dv.$$

It is to be observed that the pressure at any moment is dependent upon the volume, or

$$p=f(v),$$

and before it is possible to carry the above integration any further, the nature of the function $f(v)$ must be known.

We assume that the gas obeys Boyle's Law,* and that the expansion occurs at uniform temperature; we then have

$$pv = K,$$

where K is a constant quantity dependent for its value upon the experimental conditions and the units of measurement. Consequently,

$$p = f(v) = \frac{K}{v};$$

and by substitution in equation (2),

$$L = \int \frac{K\,dv}{v}$$

(3) $$= K \log v + C.$$

When $v = v_1$, $L = 0$; for if there were no expansion, no work would be performed. Consequently,

(4) $$0 = K \log v_1 + C.$$

Subtracting (4) from (3), we have

(5) $$L = K \log \frac{v}{v_1}.$$

Putting $v = v_2$, we have

(6) $$L = K \log \frac{v_2}{v_1};$$

but $$v_1 = \frac{K}{p_1} \text{ and } v_2 = \frac{K}{p_2},$$

so that by substitution in (6)

(7) $$L = K \log \frac{p_1}{p_2}.$$

---

* This assumption is quite legitimate, for we are dealing with ideal or perfect gases, one of the definitions of which is that they are such gases as are strictly subject to Boyle's Law.

The last equation gives the work done during the expansion of the given mass of gas while the pressure sinks from the value p_1 to that of p_2.

ART. 6. **Work done in the expansion of a highly compressed gas kept at constant temperature.** If a given mass of gas is under so great a pressure that it no longer obeys Boyle's Law, the relationship between the volume and pressure has been found to be satisfactorily given by van der Waals's equation (p. 66):

$$(1) \qquad \left(p + \frac{a}{v^2}\right)(v - b) = K,$$

where a, b, and K are constants depending upon the nature of the gas and the conditions and units of measurement. Solving this equation for p, we have

$$(2) \qquad p = f(v) = \frac{K}{v - b} - \frac{a}{v^2},$$

which, when substituted in equation (2), p. 230, gives

$$(3) \qquad L = \int\left(\frac{K}{v - b} - \frac{a}{v^2}\right)dv.$$

$$(4) \qquad L = K\log(v - b) + \frac{a}{v} + C.$$

For $v = v_1$, $L = 0$, so that

$$(5) \qquad 0 = K\log(v_1 - b) + \frac{a}{v_1} + C.$$

Subtracting (5) from (4), we find

$$(6) \qquad L = K\log\frac{v - b}{v_1 - b} - a\left(\frac{1}{v_1} - \frac{1}{v}\right),$$

or, if $v = v_2$,

$$(7) \qquad L = K\log\frac{v_2 - b}{v_1 - b} - a\left(\frac{1}{v_1} - \frac{1}{v_2}\right).$$

ART. 7. **Work done in the expansion of a gas undergoing dissociation * at constant temperature.** Consider the case of a gas that dissociates on expanding so that some of its molecules break up into two others (binary dissociation). If the gas were not at all dissociated, the relation between its volume v and pressure p is given (p. 3) by

$$pv = K, \tag{2}$$

K remaining constant during the expansion. But inasmuch as the gas is dissociated, and hence contains a larger number of molecules, the pressure is greater, increasing with the augmentation of the number of molecules, for at constant temperature the pressure of a gas is directly proportional to the number of molecules. If the gas at first contained n undissociated molecules, and if the fraction x is dissociated, then $n(1-x)$ represents the number of molecules still undissociated, and $2nx$ the number formed by dissociation. Hence the actual pressure P is to the pressure p with no dissociation as the number of molecules present in the dissociated gas is to the number originally present, or

$$P : p = n(1-x) + 2nx : n = 1 + x : 1,$$

or

$$P = (1 + x)p. \tag{3}$$

* Many gases on expanding undergo an ever-increasing **dissociation**; that is, the gaseous molecule breaks up into two or more constituent molecules. When the gas dissociates into two products, the degree of dissociation is equal to the number of molecules already dissociated, divided by the number of molecules that were present before the dissociation began. If the degree of dissociation be denoted by x and the volume by v, the equation

$$kv = \frac{x^2}{1-x} \tag{1}$$

(k being a constant) has been shown to represent the facts of the case.

We found above (p. 230) that the work done up to any instant is given by

(4) $$L = \int P\,dv,$$

P being the pressure at the end of the period for which we compute the work. In the case in hand we have just found that pressure to be

$$P = (1 + x)p,$$

so that

(5) $$L = \int (1 + x)p\,dv = \int p\,dv + \int px\,dv,$$

where both p and x are functions of v.

The first integral has already been determined (p. 231). It remains therefore to find

(6) $$\int px\,dv.$$

Expressing everything in terms of x, we have

(7) $$\int px\,dv = \int px\frac{dv}{dx}dx,$$

where p on the right is now to be regarded as a function of x.

From equation (1), in footnote, p. 233, we have

$$\frac{dv}{dx} = \frac{x(2 - x)}{k(1 - x)^2}.$$

From (1) and (2), $p = \dfrac{kK(1 - x)}{x^2}.$

We have then to find

$$\int \frac{kK(1 - x)}{x^2} \cdot x \cdot \frac{x(2 - x)}{k(1 - x)^2}dx,$$

or, simplifying,

$$\int \frac{K}{1 - x}dx + \int K\,dx,$$

with the result,

(8) $$L = -K\log(1 - x) + Kx + C.$$

We have, therefore, substituting in (5) for the integrals their values from p. 231 and (8) above, respectively,

$$(9) \qquad L = K\log\frac{v}{v_1} - K\{\log(1-x) - x\} + C.$$

When $v = v_1$, *i.e.* at the beginning of the process, no work has been done, since there has not yet been any expansion. When $v = v_1$, x will have the value x_1, which can be determined from equation (1). We have accordingly

$$(10) \qquad 0 = -K\{\log(1-x_1) - x_1\} + C.$$

Subtracting (10) from (9),

$$(11) \qquad L = K\left\{\log\frac{v}{v_1} + x - x_1 - \log\frac{1-x}{1-x_1}\right\}.$$

In particular, the work done during the expansion from the volume v_1 to the volume v_2 is

$$(12) \qquad L = K\left\{\log\frac{v_2}{v_1} + x_2 - x_1 - \log\frac{1-x_2}{1-x_1}\right\}.$$

As remarked above, x_1 and x_2 are to be found from equation (1); their values are

$$x_1 = \frac{kv_1}{2}\left(\sqrt{1+\frac{4}{kv_1}} - 1\right)$$

and

$$x_2 = \frac{kv_2}{2}\left(\sqrt{1+\frac{4}{kv_2}} - 1\right).$$

The formula for the work done will be simpler if it is expressed in terms of x_1 and x_2. From equation (1),

$$v_1 = \frac{x_1^2}{K(1-x_1)} \text{ and } v_2 = \frac{x_2^2}{K(1-x_2)};$$

and, consequently,*

$$L = K\left(x_2 - x_1 - 2\log\frac{x_1[1-x_2]}{x_2[1-x_1]}\right).$$

---

* Formulæ 6 and 7, Appendix.

ART. 8. **Maximum average temperature of a flame.** Before taking up this problem, we determine the amount of heat given up to its environment by a body of mass m when its temperature falls from θ_2 to θ_1. We first change the meaning of W from that which it had on p. 224. There $W(\theta)$ denoted the amount of heat given off when the body cooled from the initial temperature to the variable temperature θ; here, $W(\theta)$ shall denote the amount of heat given off by the body in cooling from the variable temperature θ to the final temperature θ_1, the temperature of its environment. This is necessary since the initial temperature was known in the previous case, while here it is the quantity sought.

If the specific heat c (p. 226) were not to change with the temperature, but were to remain constant, the quantity of heat W which the body gives off in cooling would have the value

(1) $$W = mc(\theta_1 - \theta_2).$$

But this condition is frequently not fulfilled, as specific heat is a function of the temperature. Denoting by θ_0 a constant temperature arbitrarily fixed as a starting point for the comparison of specific heats at different temperatures, we may, as a rule, put

(2) $$c = \alpha + \beta(\theta - \theta_0) + \gamma(\theta - \theta_0)^2 + \delta(\theta - \theta_0)^3 + \cdots,$$

where α, β, γ, δ, $\cdots$, are constants dependent upon the particular substances under consideration, and where the number of terms of the expression to be taken is regulated by the degree of accuracy of the data. The significance of the expression may be illustrated by observing that at the temperature θ_0 all the terms but the first vanish, so that $c = \alpha$. When the temperature of the body falls from $\theta + h$ to θ, the specific heat decreases from

$$c_1 = \alpha + \beta(\theta + h - \theta_0) + \gamma(\theta + h - \theta_0)^2 + \delta(\theta + h - \theta_0)^3 + \cdots$$

to $$c_2 = \alpha + \beta(\theta - \theta_0) + \gamma(\theta - \theta_0)^2 + \delta(\theta - \theta_0)^3 + \cdots.$$

If the specific heat had the value c_1 during the cooling through the temperature interval h, the amount of heat given off would be mc_1h, while if the specific heat were c_2, the heat given off would be mc_2h. But the latter amount is too small and the former too large. Hence there must be some intermediate value c' such that $mc'h$ represents the *exact* amount of heat given off in the temperature interval. On examining the form of c_1 and c_2, we see that the condition

$$c_2 < c' < c_1$$

is satisfied if c' has the form

$$c' = \alpha + \beta(\theta + \epsilon - \theta_0) + \gamma(\theta + \epsilon - \theta_0)^2 + \delta(\theta + \epsilon - \theta_0)^3 + \cdots,$$

where $0 < \epsilon < h$. We have then as the true amount of heat given off

$$(3)\quad mh\{\alpha + \beta(\theta + \epsilon - \theta_0) + \gamma(\theta + \epsilon - \theta_0)^2 + \delta(\theta + \epsilon - \theta_0)^3 + \cdots\}.$$

Another form of expressing this amount of heat is

$$(4)\qquad W(\theta + h) - W(\theta).$$

Equating (3) and (4) and dividing through by h, we have

$$(5)\quad \frac{W(\theta + h) - W(\theta)}{h}$$
$$= m\{\alpha + \beta(\theta + \epsilon - \theta_0) + \gamma(\theta + \epsilon - \theta_0)^2 + \delta(\theta + \epsilon - \theta_0)^3 + \cdots\}.$$

Letting h approach the limit zero, and noting that ϵ approaches zero with h, we have

$$(6)\quad \frac{dW}{d\theta}$$
$$= m\{\alpha + \beta(\theta - \theta_0) + \gamma(\theta - \theta_0)^2 + \delta(\theta - \theta_0)^3 + \cdots\}.$$

This expression could be integrated directly with respect to θ; but it will be more convenient later on to deal with an integral which, like the above, is expressed in terms of $\theta - \theta_0$; we introduce, accordingly, a new variable $u = \theta - \theta_0$, integrate with respect to it, and then replace u again by its value, with the result:

$$(7) \quad W = m\left\{\alpha(\theta-\theta_0) + \frac{\beta(\theta-\theta_0)^2}{2} + \frac{\gamma(\theta-\theta_0)^3}{3} + \cdots\right\} + C.$$

If $\theta = \theta_1$, the temperature of the environment, then the amount of heat given off in cooling to that temperature will be zero. Consequently, we have

$$(8) \quad 0 = m\left\{\alpha(\theta_1-\theta_0) + \frac{\beta(\theta_1-\theta_0)^2}{2} + \frac{\gamma(\theta_1-\theta_0)^3}{3} + \cdots\right\} + C,$$

or, subtracting (8) from (7),

$$(9) \quad W = m\left\{\alpha(\theta-\theta_1) + \frac{\beta}{2}[(\theta-\theta_0)^2 - (\theta_1-\theta_0)^2] + \frac{\gamma}{3}[(\theta-\theta_0)^3 - (\theta_1-\theta_0)^3] + \cdots\right\}.$$

If in the above equation we put $\theta = \theta_2$, we evidently obtain the required amount of heat given off when the body cools down from θ_2 to θ_1.

The following simple considerations will now give us the maximum average temperature of a flame. The elevation of the temperature in the flame is occasioned by the heat of combustion of the burning substance. The heat of combustion must be equal to the amount of heat which the product of the combustion gives out in cooling from the temperature of the flame to that of the chamber in which the operation takes place and can be measured by conducting the process in a closed and isolated vessel, and noting the rise

in temperature. We know then the amount of heat that the product of combustion has given off altogether, and we know to what temperature it has cooled down, *viz.* the final temperature within the vessel, and we seek to know the temperature of the product when the cooling process began; *i.e.* in equation (9) we know W and θ_1, and we seek θ, which can accordingly be determined from this equation, as everything else in it is known, the specific heat at the temperature θ being as above,

$$c = \alpha + \beta(\theta - \theta_0) + \gamma(\theta - \theta_0)^2 + \cdots.$$

We take as an illustration the burning of carbon monoxide in pure oxygen, it being known that 28 grams of carbon monoxide unite with 16 grams of oxygen to form 44 grams of carbonic acid gas, and that in this union 67,700 units of heat are evolved. The specific heat of the carbonic acid gas referred to the molecular mass, or 44, as unit, has been found to be given by the formula

$$c = 6.5 + 0.0084\ (\theta + 273),$$

where $\alpha = 6.5$, $\beta = 0.0084$, and $\theta_0 = -273°$ C. If the final temperature in the chamber be $\theta_1 = 0°$ C., the substitution of all these values in equation (9) gives

$$67{,}700 = 6.5\,\theta + \frac{0.0084(\theta^2 + 2 \times 273\,\theta)}{2}.$$

The solution of this quadratic equation gives

$$\theta = 3205°\ \text{C}.$$

In reality, many circumstances, such as radiation, dissociation,* etc., cause the actual maximum average temperature to be less than this theoretic result.

* High temperatures as well as other reasons cause many gases to break up into other gases, generally of simpler nature.

ART. 9. **Chemical reactions in which the factors are totally converted into products.** When n kinds of molecules enter into reaction, the speed of the reaction is, according to the Law of Mass Action (p. 167), proportional to the product of their concentrations.* To simplify matters, we assume that equal numbers of molecules of each substance are present, and that the concentration of each may be denoted by a at the beginning of the reaction. Then, after the lapse of the time t, the concentration of each will be equal to $(a - x)$, x designating the amounts of the substances chemically transformed.

The speed of reaction is therefore

$$(1) \qquad \frac{dx}{dt} = k(a - x)^n,$$

where k denotes a constant.

The integration, with respect to x, of the reciprocal of this expression gives

$$(2) \qquad \frac{1}{(n-1)(a-x)^{n-1}} = kt + C.$$

When $x = 0$, $t = 0$, so that

$$(3) \qquad \frac{1}{(n-1)a^{n-1}} = C,$$

and, by subtraction,

$$(4) \qquad \frac{1}{n-1}\left(\frac{1}{(a-x)^{n-1}} - \frac{1}{a^{n-1}}\right) = kt.$$

If, for example, we put $n = 2$, then

$$(5) \qquad k = \frac{1}{t}\,\frac{x}{(a-x)a}.$$

---

* **The concentration of a gas may be defined to be the mass contained in the unit volume.**

Equation (4) holds only when $n > 1$. For the case when $n = 1$, the integration leads to a logarithmic expression, an example of which we have already met in the inversion of sugar (p. 183).

We shall now proceed to consider an example of the case where the initial concentrations of the substances are different. Let two different kinds of molecules react upon each other, and let their concentrations, when $t = 0$, be a and b, respectively. We have then, at the time t, when x molecules of each substance have reacted, the speed of reaction (cf. (1) above),

$$\frac{dx}{dt} = k(a - x)(b - x). \tag{6}$$

Integrating the reciprocal of this by the use of partial fractions, we find

$$-\frac{1}{a-b}[\log(b - x) - \log(a - x)] = kt + C. \tag{7}$$

When $x = 0$, $t = 0$, and

$$C = -\frac{1}{a-b}(\log b - \log a).$$

By subtraction,

$$\frac{1}{a-b}\log\frac{(a-x)}{(b-x)}\frac{b}{a} = kt. \tag{8}$$

If in equation (8) we put $a = b$, equation (5) should result; but we encounter here the peculiar difficulty that when $a = b$, the first factor of (8) assumes the form $\frac{1}{0}$ while the logarithmic expression reduces to $\log 1$; that is, to zero. This difficulty is merely apparent, and will be cleared up in Chap. X.

ART. **10. Reactions in which the factors are only partially converted into the products.** In a reaction occurring in a homogeneous mixture of gases or a homogeneous solution at constant temperature, the Law of Mass Action states the following for the case in which the original reacting substances are not wholly used up before the reaction stops: The speed of reaction at any moment is equal to the product of the concentrations of the reacting substances minus the product of the concentrations of the substances formed by the reaction, each product being multiplied by a factor of proportionality.

Expressed in a formula, the above law becomes

$$(1)\quad \frac{dx}{dt}=k(a-x)(b-x)(c-x)\cdots-k'(a'+x)(b'+x)(c'+x)\cdots,$$

where x represents the number of molecules that have reacted in the time t, and a, b, c, $\cdots$ are the initial concentrations (corresponding to the time $t=0$), a', b', c', $\cdots$ the initial concentrations of the substances formed, and k and k' are the constants of the reaction.

At the time t, the concentrations of the reacting substances are $a-x$, $b-x$, $c-x$, $\cdots$, while those of the substances formed in the reaction are $a'+x$, $b'+x$, $c'+x$, $\cdots$.

Equation (1) is integrable, for the expression

$$\frac{1}{k(a-x)(b-x)(c-x)\cdots-k'(a'+x)(b'+x)(c'+x)\cdots}$$

can be decomposed into partial fractions (p. 203).* Such cases as have as yet been experimentally studied are very

* The integration may require the decomposition of the denominator into factors of the first or the second degree. While this is theoretically possible (*i.e.* such factors exist), it may not always be possible actually to find them by processes of algebra.

simple, the number of reacting substances never exceeding three.

ART. **11. Formation of lactones.** We take up one example illustrating what has just preceded. Certain organic acids, when dissolved in water, form compounds known as lactones. If a be the initial concentration of the acid, and a' that of the lactone, then, according to equation (1), p. 242,

$$\frac{dx}{dt} = k(a - x) - k'(a' + x), \tag{1}$$

or, taking the reciprocal, and integrating with respect to x,

$$\frac{1}{k + k'} \log [(ka - k'a') - (k + k')x] = t + C. \tag{2}$$

Since t and x vanish together,

$$-\frac{1}{k + k'} \log (ka - k'a') = C, \tag{3}$$

and, by subtraction,

$$\log \frac{ka - k'a'}{(ka - k'a') - (k + k')x} = (k + k')t. \tag{4}$$

At the expiration of a very long time, the system comes into a state of equilibrium; *i.e.* no further reaction takes place. In that case, let the concentrations of the acid and lactone be A and A', respectively. Since, when equilibrium ensues, the speed of the reaction becomes zero, equation (1) assumes the form

$$0 = kA - k'A', \tag{5}$$

or

$$\frac{k}{k'} = \frac{A'}{A} = K, \tag{6}$$

K is called the constant of equilibrium, and may be considered to be known in any given case, since it can be found experimentally. By dividing numerator and denominator of the fraction in equation (4) by k', we obtain

$$\frac{1}{t} \log \frac{Ka - a}{(Ka - a') - (1 + K)x} = k + k'. \tag{7}$$

The relationship is now in a form that may be tested by experiment. Since $(k + k')$ is a constant, the left member must also be constant. To give a numerical example, it was found by experiment that the initial

concentration of acid and lactone was $a = 18.23$ and $a' = 0$, respectively, and that

$$K = \frac{A'}{A} = \frac{13.28}{4.95} = 2.68.$$

For corresponding values of t and x, the following values for equation (7) were computed:

t	x	$k + k' = \frac{1}{t} \log \frac{Ka}{Ka - (1 + K)x}$
21	2.39	0.0350
36	3.70	0.0355
50	4.98	0.0370
65	6.07	0.0392
80	7.14	0.0392
120	8.88	0.0375
160	10.28	0.0376
220	11.56	0.0371
320	12.57	0.0357

The constancy of the numbers in the last column is quite satisfactory, and indicates that the assumptions made in developing the theory were correct in this case.

CHAPTER VIII

DEFINITE INTEGRALS

ART. 1. **The quadrature* of the parabola.** Let it be required to find the area of a segment of a parabola which is cut off by a straight line PP' (Fig. 48) perpendicular to the axis of the parabola. The axis divides this segment into two equal parts, either of which we denote by S. If we draw a tangent to the parabola at its vertex O, and let fall upon it from P the perpendicular PQ, the area of the segment S is equal to the difference between the area of the rectangle $OQPL$ and that of the figure OPQ bounded by the parabola and its tangent. We need, therefore, to determine the area of OPQ only.

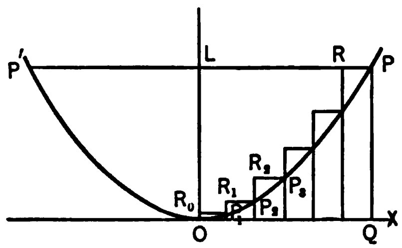

FIG. 48.

Here again we begin with a method of approximations. We take the tangent at the vertex of the parabola as the axis of abscissæ, and divide OQ, which we denote by a, into n equal parts of length h; at the points of division we erect perpendiculars, and at the points P_1, P_2, P_3, $\cdots$, where they cut the parabola, we draw lines parallel to the axis of x, so

* To find the area of a curve or a bounded portion of a plane is tantamount to finding the area of an equivalent *square*.

that they may intersect the perpendiculars at the points R_0, R_1, R_2, R_3, $\cdots$. We obtain in this way the figure

$$OR_0P_1R_1P_2R_2P_3 \cdots RPQ,$$

which is bounded by straight lines, and contains the surface whose area we have to find.

We can determine the area of this figure as follows: Since the axis of ordinates is the axis of symmetry of the parabola, its equation reads (p. 20),

$$(1) \qquad x^2 = 2\,py, \text{ or } y = \frac{x^2}{2\,p}.$$

If x_1y_1, x_2y_2, x_3y_3, $\cdots$, $x_ny_n(x_n = a)$ are the coördinates of the points P_1, P_2, P_3, $\cdots$, P, the small rectangles lying between two successive ordinates have areas equal to

$$y_1h,\ y_2h,\ y_3h,\ \cdots,\ y_nh,$$

and their total area is equal to

$$(2) \qquad \begin{aligned} A &= y_1h + y_2h + y_3h + \cdots + y_nh \\ &= h(y_1 + y_2 + y_3 + \cdots + y_n). \end{aligned}$$

We have

$$(3) \qquad x_1 = h,\ x_2 = 2\,h,\ x_3 = 3\,h,\ \cdots,\ x_n = nh = a,$$

and according to (1),

$$y_1 = \frac{h^2}{2\,p},\ y_2 = \frac{(2\,h)^2}{2\,p},\ y_3 = \frac{(3\,h)^2}{2\,p},\ \cdots,\ y_n = \frac{(nh)^2}{2\,p},$$

and by substitution in (2),

$$(4) \qquad \begin{aligned} A &= \frac{h}{2\,p}(h^2 + 2^2h^2 + 3^2h^2 + \cdots + n^2h^2) \\ &= \frac{h^3}{2\,p}(1 + 2^2 + 3^2 + \cdots + n^2), \end{aligned}$$

or,*

$$(5)\quad A = \frac{h^3}{2p}\,\frac{n(n+1)(2n+1)}{1\cdot 2\cdot 3} = \frac{hn(hn+h)(2hn+h)}{1\cdot 2\cdot 3\cdot 2p}.$$

But by (3), $nh = a$; hence,

$$A = \frac{a(a+h)(2a+h)}{1\cdot 2\cdot 3\cdot 2p}$$

$$(6)\quad = \frac{a^3}{6p} + h\frac{a^2}{4p} + h^2\frac{a}{12p}.$$

The area of the figure bounded by straight lines approaches the nearer to that of the parabola, the smaller the quantity h is taken; that is, the greater the number, n, of the rectangles, becomes. Accordingly, if h be taken small enough the area of the figure will approach just as nearly as we please to the area of OPQ. We see then that the area of OPQ is the limit which the area of the polygon approaches as h approaches zero. Denoting the area of OPQ by F, we have, therefore,

$$(7)\quad F = \lim_{h \doteq 0}\left[\frac{a^3}{6p} + h\frac{a^2}{4p} + h^2\frac{a}{12p}\right],$$

or

$$(8)\quad F = \frac{a^3}{6p}.$$

The area of the segment of the parabola itself may now be found as follows. The rectangle $OQPL$ has the sides $OQ = a$ and $PQ = y_n$. Its area is hence ay_n, and may be expressed in terms of a by equation (1) with the result

$$ay_n = \frac{a^3}{2p}.$$

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* Formula 53, Appendix.

The area of the segment of the parabola, S, that is bounded by PL and the axis of ordinates of the parabola, is

$$(9) \qquad S = \frac{a^3}{2p} - \frac{a^3}{6p} = \frac{a^3}{3p};$$

therefore,

$$(10) \qquad S = 2F.$$

Accordingly, *the parabola divides the rectangle into two parts, one of which is twice the other.*

In the problem just solved we find again confirmed that which was said, p. 101. Even at those decisive points where our conceptions lose their definiteness and become obscure, our calculations lose nothing in definiteness and clearness; they furnish what our conceptions are unable to furnish. An analogous state of affairs occurs in every case where we have to determine a sum, the number of whose parts is increasing without bound while the magnitude of each separate part approaches zero. The area of a surface of any shape, the contents of a body with variable cross-section, the total mass of a body of varying density, the sum of all the attracting forces that are exerted on a point by all the parts of a body, — all these are examples of the class of problems that may be handled in the way just set forth.

ART. 2. **Notation of sums.** For the sum

$$y_1h + y_2h + y_3h + \cdots,$$

occurring in equation (2), p. 246, the abbreviation

$$\Sigma(yh) \text{ or } \Sigma yh$$

has been adopted, where Σ indicates that a sum is to be formed of terms like yh, in which all the values of y, as

$y_1, y_2, y_3, \cdots, y_n$, are substituted in order. Another customary notation is to write Δx instead of h to bring out the fact that h is the difference of the abscissæ for two successive points of division. We have, accordingly,

$$(1) \qquad A = \Sigma yh \text{ and } A = \Sigma y\Delta x,$$

as expressions for the sum of all the rectangles. The limit of either, when h (or Δx) approaches the limit zero, is the area, F, of the segment of the parabola, *i.e.*

$$(2) \qquad F = \lim_{h \doteq 0} \Sigma yh, \text{ or } F = \lim_{\Delta x \doteq 0} [\Sigma y\Delta x].$$

We shall now prove that this limiting value may be found by substituting $x = a$ in the value of

$$(3) \qquad \int ydx$$

(or, speaking more exactly, in one of the boundless number of values which, as we know, this integral has). For if we substitute for y in (3) its value given in (1), p. 246, we find

$$(4) \qquad \int y\,dx = \int \frac{x^2}{2p}\,dx = \frac{1}{2p}\int x^2\,dx$$
$$= \frac{1}{2p}\frac{x^3}{3} + C = \frac{x^3}{6p} + C,$$

and if we take the particular case of this in which $C = 0$, and in it put $x = a$, we get the same expression that we found in (8), p. 247, for the area F.

ART. 3. **The quadrature of any curve.** What precedes can readily be extended to any curve whatever. Let

$$(1) \qquad y = f(x)$$

be the equation of the curve. We assume that the curve intersects the axis of ordinates, as is actually the case in

Fig. 49, and proceed to calculate the area of the surface bounded by the axis of x, that of y, the curve, and the ordinate PQ. As in the previous example, we imagine the abscissa OQ (which we denote by a) to be divided up into any number of small parts, which are not necessarily equal, and denote them by h_1, h_2, h_3, ... We then conceive of perpendiculars being erected at the points of division and cutting the curve at P_1, P_2, P_3, ..., and consider the figure bounded by the straight lines. Its area approximates to that of the given surface. Although a portion of this figure projects above the curve and a portion lies below it, that is of no consequence in our results. If the coördinates of P_1, P_2, P_3, ..., be denoted by

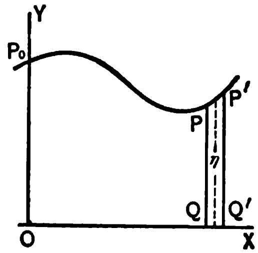

FIG. 49.

$$x_1y_1,\quad x_2y_2,\quad x_3y_3,\quad \cdots,$$

the area A of this surface is

(2) $$A = h_1y_1 + h_2y_2 + h_3y_3 \cdots;$$

or, if we put

(3) $$h_1 = x_1 - x_0 = \Delta x_1,\quad h_2 = x_2 - x_1 = \Delta x_2 \cdots,$$

then $$A = y_1\Delta x_1 + y_2\Delta x_2 + \cdots,$$

and, on using the notation of sums,

(4) $$A = \Sigma yh = \Sigma y\Delta x.$$

We know that this sum still represents the area of the surface when all the h's or the Δx's approach zero; the sum has, therefore, a definite limiting value when h approaches zero, *viz.* the area of the surface P_0OQP; and we have, letting $\lim A = F(a)$,

(5) $$F(a) = \lim\,[\Sigma y\,\Delta x].$$

We shall prove that this limit is also represented by one of the values of the integral

(6) $$\int y\,dx = \int f(x)\,dx$$

when in it we put $x = a$. To do this we must show that $F(x)$ is one of the values of $\int f(x)\,dx$; *i.e.* that $\frac{dF(x)}{dx} = f(x)$.

In order to obtain the derivative of $F(x)$, we have to find the limit of

$$\frac{F(x+h) - F(x)}{h},$$

as h approaches zero.

To this end, we change the notation slightly, and let the point Q now be a *variable* point, and denote the distance OQ by x. We also designate by P' the point in the curve that corresponds to the abscissa $x + h = OQ'$, so that

$$F(x+h) = OP_0P'Q',$$

$$F(x) = OP_0PQ,$$

and, by subtraction,

$$F(x+h) - F(x) = PQP'Q'.$$

There must be between P and P' a point with the coördinates ξ and η, such that the surface $PQP'Q'$ will be equal to a rectangle with the base $QQ' = h$ and the altitude $\eta = f(\xi)$; that is,

$$F(x+h) - F(x) = h\eta = hf(\xi);$$

hence

(7) $$\frac{F(x+h) - F(x)}{h} = f(\xi).$$

If we now let h approach zero, P' (as well as the point lying between P' and P, and having the coördinates ξ and η) will approach the limit P; that is to say, ξ will approach x, and hence

$$\lim_{h \doteq 0} \frac{F(x+h) - Fx}{h} = f(x), \tag{8}$$

or,

$$\frac{dF(x)}{dx} = f(x), \tag{9}$$

hence,

$$F(x) = \int f(x)\, dx, \tag{10}$$

which was to be shown.

Combining (5) and (10), and remembering that the constant abscissa, a, has been replaced above by the variable abscissa, x, we find

$$F(x) = \int y\, dx = \lim \{\Sigma y\, \Delta x\}, \tag{11}$$

and this equation states that the integral of the function y is nothing other than the limit which this sum approaches when the parts h or Δx into which the x-axis is divided approach the limit zero. From this fact the notation for integrals arose. The sign $\int$, proposed by Leibnitz in the early development of our subject, represents a form of *s* now obsolete, standing for the word *sum*, and $y\,dx$ represents the type of the terms of the sum. The portion dx of the symbol indicates which variable it is whose increment h or Δx is made to approach zero to obtain the limit in question. This variable is, of course, that with respect to which under the other definition of integral we should have to differentiate the integral in order to obtain the function under the sign.

To reconcile this geometric interpretation of the integral with its boundless number of values, we consider that, while an integral represents an area one of whose boundaries is the axis of ordinates, we are quite free in our choice of the position of this axis. Hence it is easily seen that there can be an indefinite number of values for each integral, and furthermore, that two of these values (which are functions of x) can, for equal values of x, differ only by a constant equal to the area comprised between the two axes of ordinates under consideration.

ART. 4. **Definite integrals.** We now propose to determine the area of a figure lying between *any* two ordinates of a curve, as $P_1Q_1 = b_1$ and $P_2Q_2 = b_2$ (Fig. 50, p. 255). These ordinates may have any position whatever, and we denote their abscissæ by a_1 and a_2. The surface F which they bound is the difference between surfaces bounded on the one side by the axis of ordinates, and on the other by P_2Q_2 and P_1Q_1, respectively. Their values are $F(a_2)$ and $F(a_1)$, so that

(1) $$F = F(a_2) - F(a_1).$$

For this case the notation

(2) $$F = \int_{a_1}^{a_2} y\,dx$$

has been adopted, meaning that the right member of the equation is equal to the difference $F(a_2) - F(a_1)$; that is,

(3) $$\int_{a_1}^{a_2} f(x)dx = F(a_2) - F(a_1).$$

Such an integral is termed a **definite integral**; a_1 is called its **lower limit** and a_2 its **upper limit.**

In order actually to find the value of the definite integral

as here defined, it would be necessary to find $F(x)$ such that its derivative is $f(x)$. (If the function $f(x)$ is at all complicated, this may be beyond our skill. Still it is often possible to find the value of the definite integral, even when we are unable to determine the function $F(x)$).

As illustration, we take the case of the parabola for which we have seen (p. 247) that

$$F(x)=\frac{x^3}{6p};$$

then, if a_1 and a_2 be the abscissæ of the points P_1 and P_2, respectively,

$$P_1Q_1P_2Q_2=\int_{a_1}^{a_2} y\,dx=\frac{a_2^3}{6p}-\frac{a_1^3}{6p}.$$

Since a definite integral gives the area of a surface with definite boundaries, its value must of course be a *definite number*. It must be independent of the value of the constant of integration; that is, it does not depend upon the position of the axis of ordinates. As a matter of fact, it appears clearly from the above discussion that whatever the constant is, it is both added and subtracted in forming the definite integral, and therefore disappears.

In contradistinction to the definite integral, the function $F(x)$ is called an **indefinite integral**, and the following notation is also sometimes used:

$$(4) \qquad \int_{a_1}^{a_2} y\,dx=\Big|_{a_1}^{a_2} F(x),$$

the right member as well as the left being equivalent to the difference $F(a_2)-F(a_1)$.

In words: *The definite integral is equal to the difference between the values of the indefinite integral for the upper and the lower limit of integration.*

A similar notation is customary when the symbol $\sum$ is used to express the sum of the products corresponding to a division of a fixed portion of the x-axis into parts. Hitherto we have either regarded the boundaries of the portion of the axis which was divided as understood, or we have specified them in words. They may be more conveniently indicated thus, $\sum_{a_1}^{a_2}$ (read "sum from a_1 to a_2"), a_1 being the abscissa of the end where the summation begins, and a_2 of that where it ends.

ART. 5. **The quadrature of the ellipse and of the hyperbola.** Assuming the axes in their customary position, the equation of the ellipse is

(1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

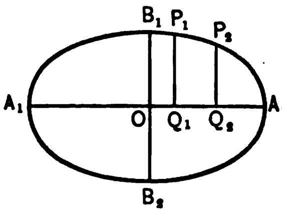

FIG. 50.

The area of the surface $P_1Q_1Q_2P_2$ (Fig. 50) is

(2) $$P_1Q_1Q_2P_2=\int_{a_1}^{a_2} y\,dx.$$

From the equation of the ellipse,

(3) $$y=b\sqrt{1-\frac{x^2}{a^2}},$$

and by substituting this value in the integral, and integrating, we obtain

$$P_1Q_1Q_2P_2=\frac{b}{2a}\Big|_{a_1}^{a_2}\left(x\sqrt{a^2-x^2}+a^2\text{ arc sin }\frac{x}{a}\right).$$

Putting $a_2=a$, $a_1=0$, we have $\frac{\pi ab}{4}$ as the area of a quadrant of the ellipse.

This result can also be obtained otherwise. We first show that the coördinates of any point of the ellipse have the values

$$x = a \cos \phi, \qquad (4)$$
$$y = b \sin \phi,$$

in which the angle ϕ is defined as follows: Constructing the auxiliary circle (p. 51) of the ellipse, and prolonging PQ to P' (Fig. 51), we denote the angle $P'OQ$ by ϕ.* In the triangle $P'OQ$,

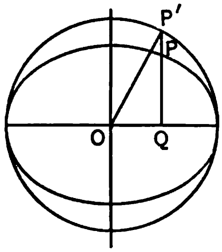

FIG. 51.

$$x = a \cos \phi,$$

and if this be substituted in equation (3), it is seen that

$$y = b \sin \phi.†$$

We have now to determine the value of the integral (2), regarded as an indefinite integral. By differentiating equation (4), we find

$$\frac{dx}{d\phi} = -a \sin \phi,$$

and

$$\int y\,dx = \int y \frac{dx}{d\phi} d\phi = -\int ab \sin^2 \phi\, d\phi = -ab \int \sin^2 \phi\, d\phi \qquad (5)$$
$$= -\frac{ab}{2} \int (1 - \cos 2\phi) d\phi ‡$$
$$= -\frac{ab}{2}\phi + \frac{ab}{4} \sin 2\phi.$$

The value of the definite integral is to be found by introducing the limits, which were originally a_2 and a_1, but we

* The angle $P'OQ$ is called the **eccentric angle** of the point P.

† Formula 20, Appendix. ‡ Formula 37, Appendix.

have since introduced a new variable ϕ, and accordingly have ϕ_2 and ϕ_1 as the values of ϕ corresponding to the points P_2 and P_1. The area E of the segment $P_1Q_1Q_2P_2$ is therefore

$$(6) \qquad E = \Big|_{\phi_1}^{\phi} \left\{ -\frac{ab}{2}\phi + \frac{ab}{4}\sin 2\phi \right\}$$

$$= \frac{ab}{2}(\phi_1 - \phi_2) - \frac{ab}{4}(\sin 2\phi_1 - \sin 2\phi_2).$$

In particular, if P_2 coincides with A_2 and P_1 with B_1,

$$\phi_1 = \frac{\pi}{2}, \qquad \phi_2 = 0;$$

hence, denoting the area of a quadrant by E_q,

$$(7) \qquad E_q = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{ab}{4}\pi.$$

Accordingly, the area of the whole ellipse is equal to $ab\pi$.

This formula is closely related to the formula for the area of a circle. The area of the auxiliary circle is $a^2\pi$; the area of the ellipse is derived from it by putting the minor semi-axis b instead of one of the factors a. This is in agreement with the fact that the ratio of any ordinate of the ellipse to that ordinate of the circle which has the same abscissa is $b : a$. (See Eq. 3, p. 51.)

We shall carry out the quadrature of the hyperbola only for the particular case that the hyperbola is equilateral; its equation, referred to its asymptotes as axes, is (p. 60)

$$(8) \qquad xy = a, \text{ or } y = \frac{a}{x}.$$

Therefore, the area H of the portion of its surface, bounded by the two ordinates y_2 and y_1 whose abscissæ are x_2 and x_1, is

$$H = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} \frac{a}{x}\,dx = a \Big|_{x_1}^{x_2} \log x$$

and (9) $$H = a(\log x_2 - \log x_1) = a \log \frac{x_2}{x_1}.$$

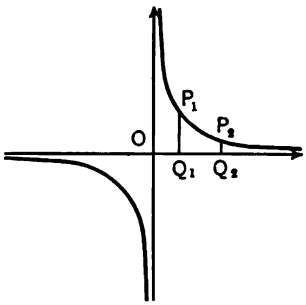

FIG. 52.

The area of any portion of the surface of an hyperbola is represented by this simple formula. If we take the lower limit at a point whose abscissa is $x_1 = 1$, we find, on denoting the upper limit by x,

(12) $$H = a \log x$$

as the area in question. On account of this relation, natural logarithms are sometimes very appropriately termed **hyperbolic logarithms.**

ART. 6. **The volume of a solid.** In order to calculate the volume of a solid, — that of the sphere, for instance, — the solid is conceived to be divided up by parallel planes into constituent parts, just as a surface was divided up by ordinates into constituent areas. A right cylinder (usually of irregular base) can be substituted for such a constituent solid, just as a rectangle was put for a constituent area; then the limit of the sum of all such cylinders, when their altitudes are made to approach zero, is the volume of the solid. This corresponds to the way in which geographic relief maps of great accuracy can be prepared by the superposition of properly cut sheets of paper.

We designate the volume of the solid by V, and divide the altitude H into a number of equal parts, each of magnitude h; through the points of division we pass parallel planes; let g denote the area of the cross-section of the solid made by one of these planes, then gh will be the volume of the right cylinder of altitude h; on g as a base; the smaller h is, the less the cylinder differs from the segment of the solid included between the same planes. The volume of the solid is the limit which the sum of all the cylinders approaches, as the number of parts into which the altitude is divided is made large without limit, and consequently each part (denoted above by h) approaches zero.

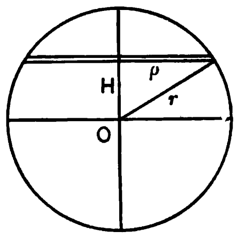

FIG. 53.

We have then

$$V = \lim_{h \doteq 0} \Sigma gh,$$

or

$$V = \int g \, dH,$$

taken between limits corresponding to the two end points of the altitude H; dH is a part of the integral symbol, because h is the increment of the altitude H.

ART. 7. **The volume of the sphere and of the paraboloid of revolution.** In the case of the sphere, the base is a small circle formed by an intersecting plane at the distance H from the center, and is equal to

(1) $$g = \rho^2\pi,$$

ρ being the radius of the small circle (Fig. 53).

But if r be the radius of the sphere,

(2) $$\rho^2 = r^2 - H^2, \text{ and } g = (r^2 - H^2)\pi,$$

whence the volume of the cylinder at that point is

(3) $$V' = \pi(r^2 - H^2)h,$$

and the volume of the sphere is

(4) $$V = \int_{-r}^{+r}(r^2 - H^2)\pi\, dH.$$

By indefinite integration,

$$\int(r^2 - H^2)\pi\, dH = \pi\left(r^2H - \frac{H^3}{3}\right),$$

and on substituting for H the upper limit r and the lower limit $-r$, the volume K of the sphere is

(5) $$K = \Big|_{-r}^{r}\pi\left(r^2H - \frac{H^3}{3}\right)$$

$$= \pi\left\{\left(r^3 - \frac{r^3}{3}\right) + \left(r^3 - \frac{r^3}{3}\right)\right\},$$

(6) $$= \frac{4\pi r^3}{3}.$$

The volume of any solid of revolution may be found in a similar way. We determine as further illustration the volume of a solid that is bounded by a surface generated by the revolution of a parabola around its axis. This surface is called the **paraboloid of revolution.** Let the equation of the parabola be $x^2 = 2py$, and let the altitude of the parabolic segment be H. We consider the volume to be the limit of the sum of circular cylinders at distance y from the vertex of the parabola, the cylinder being, accordingly, of radius x, and its thickness Δy approaching zero as a limit. We have then for the volume of one of the cylinders,

(7) $$V' = \pi x^2 \Delta y,$$

or by substituting the value of x^2,

$$V' = 2\pi py\Delta y,$$

whence

(8) $$V = \int_0^H 2\,p\pi y\,dy = \Big|_0^H y^2 p\pi,$$

or (9) $$V = H^2 p\pi.$$

The formula shows that the volume of a paraboloid of revolution is equal to that of a right cylinder with radius H and altitude p. All of its segments can, accordingly, be represented by cylinders with a constant altitude, but with a variable radius y.

ART. 8. **The mass of a rod of varying density.** *To determine the mass of a right cylindric rod whose density varies as the cube of the distance from one end.*

Let L be the length of the rod, and a the area of its cross-section. We divide the rod into n equal parts, the length of each being, accordingly, $\frac{L}{n}$, and its volume $\frac{aL}{n}$. The kth part has its nearer end at the distance $\frac{(k-1)L}{n}$, and its farther end at the distance $\frac{kL}{n}$ from the end from which we measure. The densities at the two ends of these parts are then $c\left\{\frac{(k-1)L}{n}\right\}^3$ and $c\left\{\frac{kL}{n}\right\}^3$, respectively, c being a constant.

The mass of this part, (mass equals density times volume,) is then greater than $ca(k-1)^3\left(\frac{L}{n}\right)^4$ and less than $ca\,k^3\left(\frac{L}{n}\right)^4$.

The mass of the whole rod is greater than

(1) $$\sum ca\,(k-1)^3\left(\frac{L}{n}\right)^4$$

and less than

(2) $$\sum ca\,k^3\left(\frac{L}{n}\right)^4.$$

Taking out the constant factors, we may write these sums

$$(3)\qquad caL^4\sum\frac{(k-1)^3}{n^4}\quad\text{and}\quad caL^4\sum\frac{k^3}{n^4},$$

where k assumes the values 1, 2, 3, $4\cdots n$ (since we have the sum of all the n parts). If the number of parts is increased without bound, the limit which either of the above expressions approaches is the mass sought.

We may write

$$(4)\qquad \sum\frac{(k-1)^3}{n^4}\quad\text{as}\quad\sum\left(\frac{k-1}{n}\right)^3\cdot\frac{1}{n},$$

$$\frac{k-1}{n}\text{ takes the values } 0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n}\cdots\frac{n-1}{n};$$

i.e. we have the interval from zero to unity divided into n equal parts. Putting as usual, $\frac{1}{n}=h$, we have to determine

$$(5)\qquad \lim_{h\doteq 0}\sum x^3h.$$

But this is, by the definition of definite integrals,

$$(6)\qquad \int_0^1 x^3dx.$$

The indefinite integral is $\frac{x^4}{4}$, and the value of the definite integral is, accordingly, $\frac{1}{4}$.

The mass of the rod, which was found to be caL^4 times the limit of Σ, is therefore

$$(7)\qquad \frac{caL^4}{4}.$$

ART. 9. **Some laws of operation for definite integrals.** Since definite integrals are defined as the limiting values of

sums, and may represent areas, volumes, etc., they permit of the application of certain laws of operation. An area may be divided up into parts, and the problem of finding its value may be solved by determining the area of each of its parts. Likewise, the calculation of a sum can be reduced to the calculation of the parts into which the whole sum is divided. From this self-evident principle several rules of operation are readily deduced.

If the limits of one integral are a and b, and those of another integral of the same function are b and c, then

$$(1) \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx;$$

this means simply that the area from a to c is equal to the sum of the areas from a to b and from b to c. A similar statement is true of a sum of more than two of such integrals.

A second rule is the following. According to the definition of a definite integral,

$$\int_{a_1}^{a_2} f(x)\,dx = \lim \Sigma f(x)\,\Delta x,$$

all of the quantities Δx filling up together the distance between the abscissæ a_1 and a_2, so that their sum is equal to $a_2 - a_1$. Similarly, the definite integral,

$$\int_{a_2}^{a_1} f(x)\,dx,$$

having a_1 for its upper and a_2 for its lower limit, is equal to such a limiting value, with this difference, however, the sum of all the quantities Δx must now be equal to $a_1 - a_2$, they being in this case taken with signs opposite to those of the first integral. We have then

(2) $$\int_{a_1}^{a_2} f(x)\,dx = -\int_{a_2}^{a_1} f(x)\,dx.$$

In words: *A definite integral changes sign when its limits are interchanged.*

This may also be shown in another way. We have found that

$$\int_{a_1}^{a_2} f(x)\,dx = F(a_2) - F(a_1),$$

where $F(x)$ is a function such that $\dfrac{dF(x)}{dx} = f(x)$; accordingly,

$$\int_{a_2}^{a_1} f(x)\,dx = F(a_1) - F(a_2) = -\int_{a_1}^{a_2} f(x)\,dx.$$

This law is a special case of the general mathematical principle that the opposition of positive to negative can be geometrically expressed by an opposition of direction; in this case the direction of *integration*, by which term we understand the direction in which the independent variable x increases.

In equation (1) the condition was implied that the abscissæ a, b, c were in the order of increasing magnitude. The equation is correct, however, even though this be not the case. If, for example, $a < c < b$, we have, in accordance with equation (1),

$$\int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx = \int_{a}^{b} f(x)\,dx.$$

But $$\int_{c}^{b} f(x)\,dx = -\int_{b}^{c} f(x)\,dx;$$

if we now subtract this equation from the preceding one, the remainder is

$$(3)\qquad \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx\,;$$

therefore equation (1) is true even for this case.*

EXERCISES XXVI

Find the value of the following definite integrals:

1. $\int_a^b x^3 dx.$ *Ans.* $\frac{b^4-a^4}{4}.$

2. $\int_7^{10} x^2 dx.$ *Ans.* 219.

3. $\int_0^{\frac{\pi}{2}} \cos x\,dx.$ *Ans.* 1.

4. $\int_b^a e^x dx.$ *Ans.* $e^a - e^b.$

5. $\int_0^{\pi} \cos x\,dx.$ *Ans.* 0.

6. $\int_0^1 e^{-x} dx.$ *Ans.* $1 - \frac{1}{e}.$

7. $\int_1^e \frac{1}{x} dx.$ *Ans.* 1.

8. $\int_1^3 \frac{dx}{x^3}.$

9. $\int_a^b x^9 dx.$

10. $\int_{2a}^{3a} \frac{dx}{x^2}.$

11. $\int_0^a \frac{dx}{x^2+a^2}.$

12. $\int_0^{\frac{1}{2}} \frac{x\,dx}{\sqrt{1-x^2}}.$

13. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}}.$ *Ans.* $\frac{\pi}{6}.$

14. $\int_0^1 e^x dx.$ *Ans.* $e - 1.$

15. $\int_0^1 xe^x dx.$

16. $\int_e^{e^2} \frac{a\,dx}{x}.$ *Ans.* $a.$

17. $\int_{e^r}^{e^n} \frac{dx}{x}.$ *Ans.* $n - r.$

18. $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin x\,dx.$ *Ans.* 0.

---

* We have tacitly supposed throughout that the function to be integrated does not become infinite for any value of x between the limits of integration, inclusive; (geometrically, that the curve whose area we find has no infinite branch between the limits.) In case the function becomes infinite for values of x between the limits, our results do not necessarily hold, but require further investigation, to make which would be beyond the scope of this work. We therefore presuppose in every case that the function to be integrated does not become infinite, or otherwise discontinuous, between the limits.

19. $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2 x\,dx.$ *Ans.* $\frac{\pi}{2}.$

20. $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^2 x\,dx.$ *Ans.* $\frac{\pi}{2}.$

21. $\int_0^{\frac{\pi}{4}} \tan x\,dx.$ *Ans.* $\frac{\log 2}{2}.$

22. $\int_e^{e^2} \frac{dx}{x\log x}.$ *Ans.* $\log 2.$

23. $\int_0^1 \frac{dx}{e^x + e^{-x}}$ (put $e^x = u$). *Ans.* $\arctan e - \frac{\pi}{4}.$

24. $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx.$ *Ans.* $\frac{2}{3}.$

25. The equation of the equilateral hyperbola referred to its asymptotes as axes is (p. 61)

$$xy = a^2.$$

Find the area included between the curve, the axes of x, and the ordinates $x = a$ and $x = 2a$. *Ans.* $a^2 \log 2.$

26. Find the area included between the curve $y = 5x^4$ and the x-axis from the origin to the ordinate $x = 10$. *Ans.* 100,000.

27. Find the area between the curve $y = e^x$, the axis of x, and the ordinates $x = 1$ and $x = 2$. *Ans.* $e(e-1).$

28. Show that the area of the segment of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

cut off by the ordinate at $x = c$, is

$$\frac{b}{2}\left\{\frac{c}{a}\sqrt{c^2 - a^2} - a\log\frac{c + \sqrt{c^2 - a^2}}{a}\right\}.$$

29. Find the mass of a right cylindrical rod in which the density varies as the distance from one end. *Ans.* $\frac{caL^2}{2}.$

30. Find the mass of a similar rod when the density varies as the seventh power of the distance from one end.

31. Show that the volume (**oblate spheroid**) generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about its minor axis, is $\frac{4\pi a^2 b}{3}.$

32. Show that the volume (**hyperboloid of revolution**) generated by revolving about the x-axis the arc of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

which lies in the first quadrant, and is terminated by the ordinate $x=c$, is

$$\frac{\pi b^2}{a^2}\left\{\frac{c^3}{3}-a^2c+\frac{2a^3}{3}\right\}.$$

ART. **10.** **The rectification of curves.** To find the length of an arc of a curve is equivalent to finding a straight or *right* line of equal length, into which the arc could be straightened out. The process is therefore called **rectifying** the curve.

Let it be required to find the length of the curve $y=f(x)$, *between the ordinate* $x=a$ *and* $x=b$.

In the figure (Fig. 54), let $OQ_1=a$ and $OQ_2=b$. Then we seek the length of the arc P_1P_2. Divide Q_1Q_2 up into n equal parts each of length Δx. Let $PR=\Delta y$, then the chord

$$P'P=\sqrt{\overline{\Delta x}^2+\overline{\Delta y}^2}$$

or $$P'P=\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot\Delta x.$$

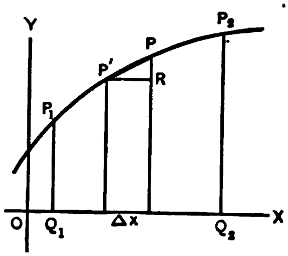

FIG. 54.

The length PR or Δy of course varies at different points along the curve, and the sum of the hypotenuses $P'P$ (corresponding to the n divisions of Q_1Q_2) is an approximation to the length of the arc. This approximation is the closer, the larger the number of divisions; *i.e.* the smaller Δx is taken. The actual length of the arc itself is

the limit which this sum approaches as Δx approaches zero; or, denoting the arc by s,

$$s = \lim_{\Delta x \doteq 0} \sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

EXAMPLE. *To find the circumference of a circle.*

The equation of the circle referred to the center as origin is

$$x^2 + y^2 = r^2.$$

Differentiating with respect to x,

$$2x + 2y\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -\frac{x}{y}.$$

We have to find then the value of

$$\int_a^b \sqrt{1 + \frac{x^2}{y^2}} dx, \text{ or of } \int_a^b \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx.$$

We find the length of one quadrant by taking the limits of the integral 0 and r, so that we have to evaluate

$$\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}} dx, \text{ or } r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = r\Big|_0^r \text{arc} \sin\frac{x}{r} = r \cdot \frac{\pi}{2}.$$

This being the length of one quadrant, the length of the whole circle is

$$2\pi r.$$

EXERCISES XXVII

1. Find the length of the curve (a **catenary**)

$$y = \frac{e^x + e^{-x}}{2}$$

from the ordinate $x = 0$ to the ordinate $x = a$. *Ans.* $s = \frac{1}{2}(e^a - e^{-a})$.

2. Find the length of the curve (a **semi-cubical parabola**)

$$y^2 = x^3$$

from the ordinate $x = 0$ to the ordinate $x = a$.

Ans. $s = \frac{8}{27}\left\{\left(1 + \frac{9a}{4}\right)^{\frac{3}{2}} - 1\right\}.$

3. Find the length of the curve (a **cycloid**)

$$y = a \operatorname{arc} \cos \frac{a - x}{a} + \sqrt{2\,ax - x^2}$$

between the same ordinates as in the previous exercises. *Ans.* $2\,a\sqrt{2}$.

4. Find the length of the curve (a **hypo-cycloid**)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

between the same ordinates as above. *Ans.* $s = \frac{3\,a}{2}$.

ART. **11. Definite and indefinite integrals.** The connection between definite and indefinite integrals, which we discussed above (p. 253), shows that the absence of an undetermined constant of integration is an advantage of calculations performed with definite integrals. Indeed, in the solution of the examples which we have treated by indefinite integrals, definite integrals might have been used from the outset. Corresponding to the indefinite integral,

$$\text{(1)} \qquad \int f(x)\,dx = F(x),$$

we have the definite integral,

$$\int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1).$$

By substituting u for $F(x)$, this equation may be written

$$\text{(2)} \qquad u_2 - u_1 = \int_{x_1}^{x_2} f(x)\,dx,$$

where u_1 and u_2 are the values of $F(x)$ corresponding to x_1 and x_2. Moreover, the differentiation of equation (1) gives

$$\text{(3)} \qquad \frac{du}{dx} = f(x).$$

In the inverse process, we pass from equation (3) to either equation (1) or equation (2), by using the method of indefinite or of definite integration.

In conclusion, we make brief illustrative application of definite integrals to some of the problems which we have already treated by use of indefinite integrals.

I. We found the equation for the inversion of sugar (p. 182) to be

$$\frac{dx}{dt} = K(a - x),$$

or

$$\frac{dt}{dx} = \frac{1}{K(a - x)}.$$

If the values t_1 and t_2 correspond to the values x_1 and x_2, the properties of definite integrals give us at once

$$t_2 - t_1 = \int_{x_2}^{x_1} \frac{dx}{K(a - x)} = \frac{1}{K} \Big|_{x_1}^{x_2} \log \frac{1}{a - x}$$

$$= \frac{1}{K} \log \frac{(a - x_1)}{(a - x_2)}.$$

II. In considering the attraction of a homogeneous rod on a point m lying in its direction, we found the following equation (p. 219):

$$\frac{dF(x)}{dx} = \frac{mM}{(a + x)^2}.$$

Now, since at one end of the rod $x = 0$ and at the other $x = l$,

$$F = \int_0^l \frac{mM\,dx}{(a + x)^2} = \Big|_0^l - \frac{mM}{a + x} = mM\left(\frac{1}{a} - \frac{1}{a + l}\right),$$

a result already found (p. 220) according to the methods of indefinite integration. The definite integral,

$$F = \int_0^l \frac{mM\,dx}{(a+x)^2}$$

gives the total attraction of the rod as the limit of the sum of the attractions exerted by each of the small constituent parts into which the rod is arbitrarily divided.

III. In like manner it appears that the altitude H above the earth's surface (p. 223) corresponding to the atmospheric pressure y is given by the definite integral,

$$H = -\int_B^y \frac{B}{S}\frac{dy}{y} = \frac{B}{S}\log\frac{B}{y}.$$

IV. Similarly (p. 226), the time elapsing during the cooling of a body from the temperature θ_1 to the temperature θ is given by the definite integral,

$$t = -\int_{\theta_1}^{\theta} \frac{mc}{K}\frac{d\theta}{\theta-\theta_0} = \frac{mc}{K}\log\frac{\theta_1-\theta_0}{\theta-\theta_0}.$$

Which of these two modes of calculation is to be chosen in any particular problem depends upon circumstances; they are in essence but slightly different, and either, correctly applied, leads to the desired result.

CHAPTER IX

HIGHER DERIVATIVES AND FUNCTIONS OF SEVERAL VARIABLES

ART. 1. **Definition of higher derivatives.** The derivative of the function

(1) $$y = \sin x$$

has the value

(2) $$y' = \frac{dy}{dx} = \cos x.$$

This derivative is likewise a function of x, and its derivative is

(3) $$\frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\sin x.$$

The expression thus obtained is called the **second derivative** (or the *second differential coefficient*) of $\sin x$, and is denoted by y'' or $\frac{d^2y}{dx^2}$. We have therefore the equation

(4) $$y'' = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\sin x = -\sin x.$$

The second derivative is also a function of x, so that we can form its derivative, and the process can be continued indefinitely. That which has just been said concerning the successive derivatives of $\sin x$ may at once be extended to all the functions which we have considered. The notation

is analogous to the preceding. Considering the function y, or $f(x)$, the expressions

$$(5) \qquad y',\ y'',\ y''',\ \cdots, \text{ or } f'(x),\ f''(x),\ f'''(x),\ \cdots,$$

or also

$$(6) \quad \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \cdots, \text{ or } \frac{d}{dx}f(x),\ \frac{d^2}{dx^2}f(x),\ \frac{d^3}{dx^3}f(x),\ \cdots,$$

denote the first, second, third, etc., derivatives of our function; in the aggregate, they are called **higher derivatives.**

Just as $\frac{d}{dx}$ has been used as a symbol for the operation of differentiating, with respect to x, that which stands after the symbol, so $\frac{d^2}{dx^2}$ is a symbol for the operation of differentiating twice with respect to x that which stands after the symbol; and, in general, $\frac{d^n}{dx^n}$ denotes the operation of differentiating with respect to x, n times. These symbols are in no sense fractions.

We have, for example

$$(7) \qquad y'' = \frac{d^2y}{dx^2} = \frac{d^2f(x)}{dx^2} = f''(x), \text{ etc.}$$

ART. 2. **The higher derivatives of the simplest functions.** I. The higher derivatives of the function e^x are the simplest of all. Putting $y = e^x$, we have (p. 144)

$$y' = e^x,\ y'' = \frac{dy'}{dx} = e^x,\ y''' = \frac{dy''}{dx} = e^x, \text{ etc.}$$

All the derivatives are therefore equal to each other and to the original function e^x itself. Here again the great simplicity of the exponential function appears clearly.

II. Considering next

$$y = \cos x,$$

we have (p. 124),

$$y' = -\sin x, \quad y'' = \frac{dy'}{dx} = -\cos x,$$

$$y''' = \frac{dy''}{dx} = \sin x, \quad y^{\text{iv}} = \frac{dy'''}{dx} = \cos x, \text{ etc.}$$

The fourth derivative is equal to the original function y; the next succeeding derivatives therefore have successively the same value as y' and the derivatives following it. The same law holds for the function $\sin x$. From $y = \sin x$ we find by repeated differentiation

$$y' = \cos x, \quad y'' = \frac{dy'}{dx} = -\sin x,$$

$$y''' = \frac{dy''}{dx} = -\cos x, \quad y^{\text{iv}} = \frac{dy'''}{dx} = \sin x.$$

The fifth derivative is equal to $\cos x$; *i.e.* equal to the first derivative, and consequently the values of the derivatives are repeated in regular sequence.

III. Taking up next

$$y = \log x,$$

we have (p. 141) $\qquad y' = \frac{1}{x} = x^{-1},$

and (p. 155) $\qquad y'' = \frac{dy'}{dx} = -1\, x^{-2} = -\frac{1}{x^2},$

$$y''' = \frac{dy''}{dx} = 1 \cdot 2\, x^{-3} = \frac{1 \cdot 2}{x^3},$$

$$y^{\text{iv}} = \frac{dy'''}{dx} = -1 \cdot 2 \cdot 3 \cdot x^{-4} = -\frac{1 \cdot 2 \cdot 3}{x^4},$$

etc., $\qquad$ etc.

IV. We consider finally

$$y = x^n,$$

where n is any arbitrary constant. We obtain (p. 155)

$$\begin{aligned}
y' &= nx^{n-1}, \\
y'' &= n(n-1)x^{n-2}, \\
y''' &= n(n-1)(n-2)x^{n-3}, \\
y^{\text{iv}} &= n(n-1)(n-2)(n-3)x^{n-4}, \\
\text{etc.,} & \qquad \text{etc.}
\end{aligned}$$

If, in particular, n is a positive integer, the exponent of x in the $(n-1)$st derivative takes the value $n-(n-1)$, or unity, and we have

$$\begin{aligned}
y^{(n-1)} &= n(n-1)(n-2)\cdots 2 \cdot x, \\
y^{(n)} &= n(n-1)(n-2)\cdots 2 \cdot 1,
\end{aligned}$$

and since $y^{(n)}$ is a constant, all the succeeding derivatives are zero. If n is not a positive integer, the sequence of the derivatives can be extended as far as we please without reaching one whose value is zero.

EXERCISES XXVIII

Find the second and the third derivative of each of the following functions:

1. $y = e^{ax}$.

2. $y = \sin ax$.

3. $y = e^x \sin x$.

4. $y = \sin^2 x$.

5. $y = \sin^3 x$.

6. $y = xe^x$.

7. $y = x^2 \log x$.

8. $y = \dfrac{1-x}{1+x}$.

9. $y = \dfrac{x^2}{1-x}$.

Find the nth derivative of the following:

10. $y = e^{ax}$.

11. $y = \sin ax$.

12. $y = (a+x)^{\frac{2}{3}}$.

13. $y = \log x^5$.

14. $y = \dfrac{1}{1+x}$.

15. $y = \dfrac{1}{\sqrt{e^x}}$.

Art. 3. **Geometric meaning of the second derivative.** The second derivative has an important geometric signification. The curve (Fig. 55) is said to be **concave** toward the x-axis in the part ABC, and **convex** toward the x-axis in the part CDE. Let

$$(1) \qquad y = f(x)$$

be the equation of the curve, and let us turn our attention to the values which $\tan \tau$ assumes along the various parts of the curve (p. 123). Beginning at A, the angle τ, and likewise $\tan \tau$, continually decreases as the point describes the arc AB. At the highest point B of the curve, the angle τ is zero, and beyond B, τ is but little less than π, and again decreases continually until the moving point reaches C; at the point B, $\tan \tau$ is zero, and beyond B it has very small and continually increasing negative values along BC; *i.e.* $\tan \tau$ decreases along BC also. For this entire arc ABC the derivative of $\tan \tau$ must therefore be negative (p. 125); that is, we must have

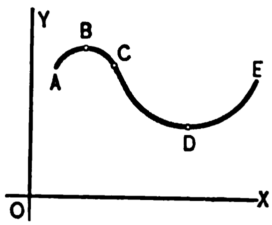

Fig. 55.

$$(2) \qquad \frac{d \tan \tau}{dx} = \frac{dy'}{dx} = \frac{d^2y}{dx^2} < 0.$$

The opposite is true for the arc CDE. From C to D, $\tan \tau$ has negative values, which decrease numerically to zero, and from D to F, $\tan \tau$ assumes continually increasing positive values. Therefore, for this part of the curve,

$$(3) \qquad \frac{d \tan \tau}{dx} = \frac{d^2y}{dx^2} > 0.$$

We see thus that the sign of the second derivative indicates whether the curve is concave or convex toward the x-axis.

The two arcs of the curve are separated by the point C, in which the second derivative becomes

$$\frac{d^2y}{dx^2} = 0; \tag{4}$$

and this is called a **point of inflexion.**

The convexity or concavity of a curve naturally depends upon the side from which the curve is viewed. As our figure was constructed above, the curve lay entirely on the positive side of the x-axis. If the curve, on the other hand, lies entirely on the negative side of the x-axis (the x-axis may be imagined to be moved parallel to itself until this is the case), the part of the curve which was formerly convex toward the x-axis is now concave, and *vice versa*. To obtain a complete criterion for the direction of curvature of the curve, we notice the following: In Fig. 55 the arc ABC is concave toward the x-axis; for it, y is positive and $\frac{d^2y}{dx^2}$ is negative; in the altered position of the x-axis, CDE is concave toward the x-axis; for it, y is negative and the second derivative positive; in both cases, therefore,

$$y\frac{d^2y}{dx^2} < 0. \tag{5}$$

For convex arcs the reverse is true. Along the arc CDE in Fig. 55, both y and the second derivative are positive, while in the altered figure both these quantities are negative for the arc ABC, and therefore in both cases

$$y\frac{d^2y}{dx^2} > 0. \tag{6}$$

We have thus obtained a complete criterion to determine whether any arc of a curve is convex or concave toward the x-axis.

A simple example is offered by the sine curve (Fig. 39, p. 123), whose equation is

$$y = \sin x.$$

The second derivative is alternately negative or positive, while the corresponding y is alternately positive or negative; the curve is, therefore, always concave toward the x-axis. The intersections of the curve with the x-axis are all points of inflexion.

If we select the point of view below the curve (the direction in which a point with constant abscissa moves when its ordinate is diminished, being regarded as downward), we may also say: If $\frac{d^2y}{dx^2}$ is positive for any value of x, the curve is convex downward for that abscissa, while if $\frac{d^2y}{dx^2}$ is negative, the curve is concave downward.*

ART. 4. **Physical interpretation of the second derivative.** Like the first derivative, the higher derivatives also are of great importance in the applications to physical science. For our purposes it will suffice to make the meaning of the second derivative clear by several examples.

Let any rectilinear motion (*e.g.* that of a freely falling body) be given by the equation

(1) $$l = f(t).$$

Suppose that in $t, t_1, t_2, t_3, \cdots$

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* An exception *may* arise when $\frac{dy}{dx} = 0$ or ∞ for the value of x in question. This case will be taken up in another connection (p. 362).

units of time (*e.g.* seconds) from the beginning of the movement, the moving point has passed over

$$l,\ l_1,\ l_2,\ l_3,\ \cdots$$

units of length (*e.g.* meters), respectively, and let its velocity at these times be, respectively,

$$v,\ v_1,\ v_2,\ v_3,\ \cdots.$$

The velocity will, in general, change each instant. To form an idea of the nature of this change, we introduce the well-known concept of *acceleration.* Considering first *uniformly accelerated motion, i.e.* motion in which the velocity receives equal increments in equal intervals of time, *the* **acceleration** *is the increase in velocity which takes place in a second.* If in τ seconds the increase in velocity has been η meters, and if α denote the acceleration, we have

$$\alpha = \frac{\eta}{\tau}. \tag{2}$$

We now apply this idea to the motion represented by equation (1), which is not uniformly, though continuously, accelerated, *i.e.* the increase in velocity is not always the same in equal intervals of time at different periods in the motion, but there is no instantaneous increase due to the impetus of a new force suddenly applied. As there is no perceptible instantaneous increase at any point in the motion, the amounts of increase in velocity in consecutive time intervals approach more and more nearly to equality as the intervals of time are taken smaller and smaller, and the limit which the ratio of the time interval to the increment of velocity approaches as the former is diminished, is by (2) the acceleration. If Δv denote the increase in velocity in the time Δt, then the limit of the ratio $\dfrac{\Delta v}{\Delta t}$ as Δt

approaches zero is the acceleration. But this limit is the derivative of v with respect to t, and hence

$$(3) \qquad \alpha = \frac{dv}{dt}.$$

We have seen in (p. 109), that v is the derivative of l with respect to the time, *i.e.*,

$$(4) \qquad v = \frac{dl}{dt},$$

and hence

$$(5) \qquad \alpha = \frac{dv}{dt} = \frac{d^2l}{dt^2}.$$

For instance, we have for the motion of free fall (p. 167),

$$v = gt,$$

and hence the acceleration is

$$\frac{d^2l}{dt^2} = \frac{dv}{dt} = g.$$

This result shows that the acceleration is a measure of the force which causes the motion; and in fact, we may *define* forces by means of the accelerations which they impart in the unit of time to a body of unit mass on which they act. If, therefore, the law of any motion is known, we can determine what the forces are which cause the motion.

Art. 5. **Oscillatory motion.** Let, for example, a point P of unit mass move upon a straight line (Fig. 56) so that its distance x from a fixed point O is given by the equation

Fig. 56.

$$(1) \qquad x = A \sin t.$$

For the velocity of the point we have

$$(2) \qquad v = \frac{dx}{dt} = A \cos t,$$

and from this by forming the second derivative we have for the acceleration

$$(3) \qquad \alpha = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A \sin t.$$

Substituting for $A \sin t$ its value x from equation (6), we have

$$(4) \qquad \frac{d^2x}{dt^2} = -x.$$

This equation states that the acceleration, and therefore *the force acting on the point, is equal to its distance from the fixed center.* The negative sign denotes that the tendency of the force is to diminish the distance of the movable point from O; that is, the force is attractive.

Under this law are included most of the motions in which bodies oscillate about a center of equilibrium. Such are, according to Huygens, the motion of the particles of ether in the propagation of light, the motion of the particles of air in the propagation of sound, the vertical motion of the particles of water in the propagation of water-waves, the movement of the particles of a vibrating string; in short, all those movements of small particles which take place in the motion of either stationary or progressive waves.

The nature of this motion is like that of the pendulum about its position of rest. If for t we put, respectively,

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi, \quad \frac{5\pi}{2}, \quad \cdots,$$

then we have for x, v, α, at the times just mentioned, the values

$$\begin{aligned} x &= 0, \quad A, \quad 0, \quad -A, \quad 0, \quad A, \quad \cdots; \\ v &= A, \quad 0, \quad -A, \quad 0, \quad A, \quad 0, \quad \cdots; \\ \alpha &= 0, \quad -A, \quad 0, \quad A, \quad 0, \quad -A, \quad \cdots. \end{aligned}$$

These results enable us to sketch the progress of the motion. At the beginning of the motion, *i.e.* at the time $t = 0$, the point is at O and has the velocity A, while the active force is zero. After $\frac{\pi}{2}$ seconds the point is at N, its velocity has in the meantime been reduced to zero, while the attractive force has reached its greatest value, A. The point now reverses the direction of its motion, and in consequence of the attractive force its velocity increases constantly. In π seconds the point is again at O, and passes through O with the highest velocity that it can attain. In $\frac{3\pi}{2}$ seconds the point is at M, and has attained its greatest distance from O toward the left; its velocity is again zero, while the force has again reached its maximum value. In 2π seconds the point is again at O, passes through O with the velocity A, and the attractive force is zero. The latter now increases again, while the velocity constantly diminishes until the point reaches N again, and thus it continually and regularly swings back and forth in a straight line.

ART. 6. **The velocity acquired by a body falling toward the earth from a great distance.** We suppose that the body falls from rest, under the influence of the earth's attraction alone. As the body falls from a great distance, we may not suppose, as heretofore, that the acceleration is constant, but must take into account the law that the acceleration varies inversely as the square of the distance between the attracting bodies. Let x and x_0 denote the distance of the body from the earth's surface at the time t and at the time the fall began, respectively, and let g be the constant of gravity at the earth's surface, and r the radius of the earth. Then, as we have already seen (p. 109), the velocity at this time is

(1) $$v = \frac{d(x_0 - x)}{dt} = -\frac{dx}{dt},$$

and the acceleration is

(2) $$\alpha = -\frac{d^2x}{dt^2}.$$

Multiplying both members of (2) by $-2\frac{dx}{dt}$, and integrating with respect to t, we have

(3) $$\int -2\,\alpha\frac{dx}{dt}\,dt = \left(\frac{dx}{dt}\right)^2 = v^2.$$

The velocity and the acceleration are only apparently negative since the variable x is decreasing, and hence the first derivative is itself negative, while by its definition, V is increasing. The acceleration at the earth's surface is g, and by Newton's law of gravitation,

(4) $$\frac{\alpha}{g} = \frac{r^2}{x^2}.$$

From (3) and (4),

(5) $$\int -\frac{2\,gr^2}{x^2}\frac{dx}{dt}\,dt = v^2.$$

(6) But $$\int -\frac{2\,gr^2}{x^2}\frac{dx}{dt}\,dt = \int -\frac{2\,gr^2}{x^2}\,dx = \frac{2\,gr^2}{x} + C,$$

or,

(7) $$v^2 = \frac{2\,gr^2}{x} + C.$$

When the fall began, $v = 0$, so that

$$0 = \frac{2\,gr^2}{x_0} + C,$$

whence,

(8) $$v^2 = 2\,gr^2\left(\frac{1}{x} - \frac{1}{x_0}\right).$$

This formula gives the velocity at the distance x from the earth's center. At the earth's surface,

$$(9) \qquad v^2 = 2\,gr^2\left(\frac{1}{r} - \frac{1}{x_0}\right).$$

If the distance x_0 is increased, v^2 approaches the limit $2\,gr$. No matter from what distance a body falls to the earth (under the influence of the earth's attraction alone), its velocity on reaching the earth will always be less than $\sqrt{2\,gr}$, which may readily be found to be not quite 7 miles per second. Disregarding the resistance of the earth's atmosphere, we see that if a body were projected vertically upward with the initial velocity, $\geqq \sqrt{2\,gr}$ (*e.g.* seven miles per second), it would never fall back to the earth.

In a similar manner, the limit of the velocity with which a falling body would reach other bodies can be determined. For the sun, the velocity is about 383 miles per second. A body passing the earth's orbit, under the influence of the sun only, would be moving at about 26 miles per second. This is nearly the average velocity with which meteors enter our atmosphere.

The velocity needed to project a body beyond the range of the moon's attraction is small; it has been supposed that the moon lost its atmosphere for this reason. For the sun, on the other hand, the velocity is very great, and the sun retains an enormous atmosphere.

The formula (9) can also be used in fall through a short distance if we wish to take into account the variation of the earth's attraction, since we introduced no hypothesis as to the magnitude of x_0 until after (9) was established. For a small distance the difference between the velocity computed by (9) and that found on the assumption that the earth's attraction remained constant would, of course, be very slight.

ART. 7. **Partial derivatives.** If a gas is subjected to a variable pressure and temperature, its volume v is dependent upon the pressure p as well as the temperature θ. We say, therefore, that v is a function of p and of θ. Similarly, the area of an ellipse is a function of its semi-axes, a and b; the volume of a parallelopiped is a function of its three

edges, etc. These are examples of *functions of two or more variables*. In correspondence with our previous nomenclature (p. 112), we call the volume v, regarded as a function of p and θ, the *dependent* variable, while we call p and θ the two *independent* variables; similarly in the other instances mentioned.

Denoting the variables by $x, y, z, \cdots$, the following symbols have been introduced for functions of two or more variables:

$$f(x, y),\ F(x, y, z),\ \phi(x, y), \cdots, \text{ etc.}$$

We now extend the rules and theorems of the differential calculus to these functions.

We begin with the following illustration: Let S be the area of the rectangle $OABC$ (Fig. 57), whose sides have the lengths x and y, so that

(1) $$S = xy.$$

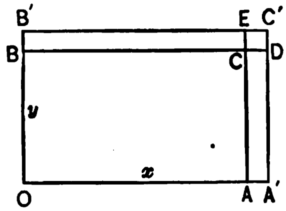

FIG. 57.

Regarding the lengths of these sides as variable, the area S of the rectangle will vary with them. We first vary the rectangle by altering only the length x of the side OA, and leaving the side $OB = y$ fixed. Regarded so, the area S is a function of x alone, and we can apply to it all the rules of the Differential Calculus previously explained. We assign, therefore, to the side $OA = x$ an increment AA', which we represent by Δx, and determine the corresponding derivative of S. To show that we now regard S as function of x alone, we denote this derivative by $\frac{dS(x)}{dx}$, and by differentiating equation (1) we have

(2) $$\frac{dS(x)}{dx} = y.$$

The rectangle $OABC$ can also vary so that the length of the side OB is changed, while x remains constant; the area S is then a function of y alone. We assign now to y an increment Δy, and in this case denote the derivative by $\frac{dS(y)}{dx}$; if under these assumptions we differentiate equation (1) with respect to y, we find

$$\frac{dS(y)}{dy} = x. \tag{3}$$

The notation,

$$\frac{\partial S}{\partial x} \text{ and } \frac{\partial S}{\partial y}, \tag{4}$$

has been introduced for the two derivatives above, in which the round ∂ is used to indicate that we regard S in one case as a function of the one variable x only, and in the other case as a function of the variable y only. We have, therefore,

$$\frac{\partial S}{\partial x} = y, \quad \frac{\partial S}{\partial y} = x. \tag{5}$$

What is said above for the derivative of the area of a variable rectangle may be extended to all functions of two independent variables. Let

$$u = f(x, y)$$

be such a function. We may first suppose that x varies but y remains constant, or, in other words, we may regard u as a function of x alone; if upon this hypothesis we differentiate u with respect to x, we obtain the **partial derivative** of u, with respect to x, the notations for which are

$$\frac{\partial u}{\partial x} \text{ and } \frac{\partial f(x, y)}{\partial x},$$

the round ∂'s serving to remind us constantly that we are dealing with *partial* derivatives.

Similarly, if we suppose u to be a function of y alone, we obtain, by differentiating with respect to y,

$$\frac{\partial u}{\partial y}, \text{ or } \frac{\partial f(x, y)}{\partial y},$$

the partial derivative of u with respect to y.

EXAMPLES

1. Let $u = ax^2 + cy^2$,

then $$\frac{\partial u}{\partial x} = 2\,ax,$$

and $$\frac{\partial u}{\partial y} = 2\,cy.$$

2. $u = x^2 - y^2$.

$$\frac{\partial u}{\partial x} = 2\,x; \quad \frac{\partial u}{\partial y} = -2\,y.$$

3. $u = \sin x \cos y$.

$$\frac{\partial u}{\partial x} = \cos x \cos y; \quad \frac{\partial u}{\partial y} = -\sin x \sin y.$$

4. $u = \log(x^2 + y^2)$.

$$\frac{\partial u}{\partial x} = \frac{2\,x}{x^2 + y^2}; \quad \frac{\partial u}{\partial y} = \frac{2\,y}{x^2 + y^2}.$$

5. $u = \sin \frac{x}{y}$.

To differentiate this we put $\frac{x}{y} = z$, then

$$u = \sin z,$$

and
$$\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x} = \cos z \cdot \frac{1}{y} = \frac{1}{y}\cos\frac{x}{y}.$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y} = \cos z \cdot \frac{-x}{y^2} = -\frac{x}{y^2}\cos\frac{x}{y}.$$

The above definitions and notations may be immediately extended to functions of three or more independent variables.

If $$u = f(x, y, z)$$

be a function of the three independent variables x, y, z, then we have the three partial derivatives,

$$\frac{\partial u}{\partial x}, \text{ or } \frac{\partial f(x, y, z)}{\partial x}, \qquad \frac{\partial u}{\partial y}, \text{ or } \frac{\partial f(x, y, z)}{\partial y},$$

and $$\frac{\partial u}{\partial z}, \text{ or } \frac{\partial f(x, y, z)}{\partial z},$$

in which, in each case, the definition requires that the variable with respect to which the differentiation is made, be regarded as the only one to vary.

ART. 8. **Higher partial derivatives.** Functions of two or more variables may be differentiated repeatedly with respect to any or all of the variables. The notation used is the following:

If $$u = f(x, y),$$

$\dfrac{\partial^2 u}{\partial x^2}$ denotes the result of differentiating u twice with respect to x, y being treated as a constant;

$\dfrac{\partial^3 u}{\partial x^3}$, $\dfrac{\partial^n u}{\partial x^n}$ denote the result of differentiating similarly three or n times;

$\dfrac{\partial^2 u}{\partial y\,\partial x}$ denotes the result of differentiating u first with regard to x, and then differentiating that result with regard to y;

$\dfrac{\partial^3 u}{\partial x\,\partial y^2}$ denotes the result of differentiating twice with regard to y, then with regard to x. Similar notations, understood without difficulty, are used for still more differentiations.

Let us take, for example,

(1) $$u = \sin x \cos^2 y.$$

Then $\dfrac{\partial u}{\partial x} = \cos x \cos^2 y$, $\dfrac{\partial u}{\partial y} = -2 \sin x \cos y \sin y$.

Differentiating each of these with respect to x, and also with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cos^2 y, \qquad \frac{\partial^2 u}{\partial x\,\partial y} = -2 \cos x \cos y \sin y,$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = -2 \cos x \cos y \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -2 \sin x \left[\cos^2 y - \sin^2 y\right].$$

We notice in the above illustration that the values of $\frac{\partial^2 u}{\partial y\, \partial x}$ and $\frac{\partial^2 u}{\partial x\, \partial y}$ are equal. This holds true as a general theorem.*

It may be made evident as follows :

Consider $u = f(x, y)$; we wish to prove that

$$(2) \qquad \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

By definition,

$$(3) \qquad \frac{\partial u}{\partial x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

$$(4) \qquad \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial x}\right\} = \lim_{\Delta y \doteq 0}\left\{\lim_{\Delta x \doteq 0} \frac{\overline{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)} - \overline{f(x + \Delta x, y) - f(x, y)}}{\Delta x\, \Delta y}\right\}.$$

Similarly,

$$(5) \qquad \frac{\partial u}{\partial y} = \lim_{\Delta y \doteq 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

and

$$(6) \qquad \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial y}\right\} = \lim_{\Delta x \doteq 0}\left\{\lim_{\Delta y \doteq 0} \frac{\overline{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)} - \overline{f(x, y + \Delta y) - f(x, y)}}{\Delta x\, \Delta y}\right\}.$$

A slight rearrangement will show that the fractions of which the limits are taken are the same in both cases, the only difference in the two expressions being that Δx and Δy approach zero in different orders. It seems plausible † to think, however, that the final results are the same, no matter

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* Of course, with the restriction (which we always tacitly make) that the function and all the derivatives concerned are continuous.

† The rigorous proof of this theorem is too difficult to find a place here.

in what order Δx and Δy are made zero. For in the end there are left, as the limit sought, those terms, and those only, which vanish with neither Δx nor Δy. The same limit is therefore found in each case, or

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

As an immediate consequence of this result, it is apparent that the order of any number of differentiations is immaterial. Thus, there are only four distinct third partial derivatives of u above, viz.:

$$\frac{\partial^3 u}{\partial x^3},\ \frac{\partial^3 u}{\partial y\,\partial x^2},\ \frac{\partial^3 u}{\partial y^2\,\partial x},\ \frac{\partial^3 u}{\partial y^3}.$$

All others are reducible to one or other of these by an interchange of order of differentiation; thus,

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial x} = \frac{\partial^3 u}{\partial y\,\partial x^2}.$$

EXERCISES XXIX

Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for each of the following functions:

1. $u = x^2 + y^2 - a^2$.
2. $u = x \cos y$.
3. $u = ye^x + xe^y$.
4. $u = x^3 + ax^2y + bxy^2 + y^3$.
5. $u = x^5 - 3xy^2 + 6y^4$.

HINT. — In the following four exercises use the method of p. 152.

6. $u = \sqrt{x^2 + y^2}$.
7. $u = \log(x - y^2)$.
8. $u = \sin(x^2 - y^2)$.
9. $u = e^{\frac{x^2}{y^3}}$.

Verify that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ for each of the following functions:

10. $u = x \sin y$.
11. $u = xe^x \log y$.
12. $u = \tan x^2y$.
13. $u = 5x^6 + 13x^3y^4 + 12x^2y - 32y^3$.
14. $u = \cot \frac{x+y}{x}$.
15. $u = x^2 \sin y + y \sin^2 x$.
16. $u = y^x$.
17. $u = \sin \sqrt[3]{xy^2}$.

Form $\dfrac{\partial^3 u}{\partial y^2 \partial x}$ and $\dfrac{\partial^3 u}{\partial x^3}$ for each of the following functions:

18. $u = x(x^4 + y^5)$.

19. $u = ax^2 + bxy + cy^2$.

20. $u = \sin x \cos y$.

21. $u = e^{x^2 y^3}$.

22. $u = x^y$.

23. $u = \log \dfrac{x}{y}$.

24. $u = \dfrac{x}{a + x^2 + y^2}$.

ART. 9. **Differentiation of a function of two or more functions of a single independent variable.** We have already (p. 152) considered the differentiation of a function of one function of a single variable, and have found the following result:

If $y = f(z)$ and $z = \phi(x)$, or $y = f\{\phi(x)\}$,

(1) then
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

We shall next consider the case of a function of two functions of a single variable, or of two variables *dependent* upon a single independent variable.

Accordingly, let
$$u = f(y, z),$$
where
$$y = \phi(x) \text{ and } z = \psi(x).$$

The result which we shall prove is

(2)
$$\frac{du}{dx} = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}.$$

In words: *The derivative of the given function with respect to the independent variable is equal to the partial derivative of the given function with respect to one of the dependent variables times the derivative of that dependent variable with respect to the independent variable, plus a similar product in which the other dependent variable is used.*

Letting $u+\Delta u$ be the value which u assumes when we give x the value $x+\Delta x$, then, by the definition of a derivative,

(3) $$\frac{du}{dx}=\lim_{\Delta x \doteq 0}\frac{u+\Delta u-u}{\Delta x}.$$

To find the value of this limit it is necessary to determine $u+\Delta u$ more closely.

We have $u=f(y, z)=f\{\phi(x), \psi(x)\}$.

Replacing x by $x+\Delta x$,

let $y=\phi(x)$ become $y+\Delta y$ or $\phi(x+\Delta x)$

and $z=\psi(x)$ become $z+\Delta z$ or $\psi(x+\Delta x)$,

(4) then

$$u+\Delta u=f\{\phi(x+\Delta x), \psi(x+\Delta x)\}=f(y+\Delta y, z+\Delta z).$$

Substituting this in (3),

(5) $$\frac{du}{dx}=\lim_{\Delta x \doteq 0}\frac{f(y+\Delta y, z+\Delta z)-f(y, z)}{\Delta x}.$$

Adding and subtracting $f(y, z+\Delta z)$ in the numerator,

(6) $$\frac{du}{dx}=\lim_{\Delta x \doteq 0}\frac{f(y+\Delta y, z+\Delta z)-f(y, z+\Delta z)+f(y, z+\Delta z)-f(y, z)}{\Delta x}$$

$$=\lim_{\Delta x \doteq 0}\frac{f(y+\Delta y, z+\Delta z)-f(y, z+\Delta z)}{\Delta x}$$

$$+\lim_{\Delta x \doteq 0}\frac{f(y, z+\Delta z)-f(y, z)}{\Delta x}.$$

Multiplying numerator and denominator of these fractions respectively by Δy and Δz,

$$(7) \quad \frac{du}{dx} = \lim_{\Delta x \doteq 0} \frac{f(y+\Delta y, z+\Delta z) - f(y, z+\Delta z)}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$$
$$+ \lim_{\Delta x \doteq 0} \frac{f(y, z+\Delta z) - f(y, z)}{\Delta z} \cdot \frac{\Delta z}{\Delta x}.$$

To determine the value of these limits we notice that when Δx approaches zero, Δy and Δz approach zero likewise. We may write then

$$(8) \quad \frac{du}{dx} = \lim_{\Delta y \doteq 0} \frac{f(y+\Delta y, z+\Delta z) - f(y, z+\Delta z)}{\Delta y} \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$$
$$+ \lim_{\Delta z \doteq 0} \frac{f(y, z+\Delta z) - f(y, z)}{\Delta z} \lim_{\Delta x \doteq 0} \frac{\Delta z}{\Delta x}.$$

The various factors here are all in the defining form for derivatives except the first, *viz.*:

$$(9) \qquad \lim_{\Delta y \doteq 0} \frac{f(y+\Delta y, z+\Delta z) - f(y, z+\Delta z)}{\Delta y}.$$

This would be in the defining form for a derivative if Δz were constant; namely, it would be $\frac{\partial}{\partial y} f(y, z+\Delta z)$. But under the conditions of our problem, Δz approaches zero at the same time that Δy does so, so that $\lim_{\Delta y \doteq 0} (z+\Delta z)$ is z, and hence

$$(10) \quad \lim_{\Delta y \doteq 0} \frac{f(y+\Delta y, z+\Delta z) - f(y, z+\Delta z)}{\Delta y} = \frac{\partial}{\partial y} f(y, z).$$

We use the round ∂'s to indicate that $f(y, z)$ is differentiated with respect to y alone.

Substituting this in (8), and replacing the other factors by their limits, we obtain, finally,

$$(11) \qquad \frac{du}{dx} = \frac{\partial u}{\partial y} \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx}.$$

In quite the same way, we can establish similar formulæ for functions of more than two functions of an independent variable. Thus, if $u = f(y, z, w)$, y, z, and w, being functions of x, the method used above will show that

$$(12) \qquad \frac{du}{dx} = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx} + \frac{\partial u}{\partial w}\frac{dw}{dx}.$$

EXAMPLES

1. Let
$$u = \frac{z^2}{a^2} + \frac{y^2}{b^2},$$
where
$$z = a^2b^2 \cos x,$$
$$y = a^2b^2 \sin x;$$
then
$$\frac{\partial u}{\partial y} = \frac{2y}{b^2}; \qquad \frac{\partial u}{\partial z} = \frac{2z}{a^2};$$
$$\frac{dy}{dx} = a^2b^2 \cos x; \quad \frac{dz}{dx} = -a^2b^2 \sin x;$$
and,
$$\frac{du}{dx} = \frac{2y}{b^2} \cdot a^2b^2 \cos x + \frac{2z}{a^2}(-a^2b^2 \sin x)$$
$$= 2a^2y \cos x - 2b^2z \sin x$$
$$= 2a^2b^2 \sin x \cos x (a^2 - b^2).$$

2. Let
$$u = (ay^2 + bz^2)^n,$$
where
$$y = e^x; \quad z = e^{-x};$$
then
$$\frac{\partial u}{\partial y} = n(ay^2 + bz^2)^{n-1} \cdot 2ay;$$
$$\frac{\partial u}{\partial z} = n(ay^2 + bz^2)^{n-1} \cdot 2bz;$$
$$\frac{dy}{dx} = e^x; \quad \frac{dz}{dx} = -e^{-x}.$$
Hence
$$\frac{du}{dx} = 2anye^x(ay^2 + bz^2)^{n-1} - 2bnze^{-x}(ay^2 + bz^2)^{n-1}$$
$$= 2ne^{2x}(ay^2 + bz^2)^{n-1}(a - be^{-4x}).$$

The same result would, of course, be reached if we first substitute the values of y and z in u, and then differentiate.

3. $$u = x^x.$$

This function involves only the one variable x, and we have found its derivative before by passing to logarithms, inasmuch as none of our fundamental rules for differentiation were directly applicable. We can also differentiate it by our present method, regarding it as given thus:

$$u = y^z,$$

where

$$y = x,$$

$$z = x;$$

$$\frac{\partial u}{\partial y} = zy^{z-1}; \quad \frac{\partial u}{\partial z} = y^z \log y;$$

$$\frac{dy}{dx} = 1; \quad \frac{dz}{dx} = 1.$$

Hence $\dfrac{du}{dx} = zy^{z-1} + y^z \log y$, or substituting the values of y and z,

$$\frac{du}{dx} = x^x + x^x \log x = x^x(1 + \log x).$$

EXERCISES XXX

Find $\dfrac{du}{dx}$ for the following functions:

1. $u = z^3 + y^3$.
$z = \sin x;\ y = \cos x$.

2. $u = z \log y$.
$y = \sqrt{x^3};\ z = e^x$.

3. $u = z^y$. $y = \log x;\ z = x^2$.

4. $u = z^2 + yz + y^2$.
$z = e^x;\ y = e^{-ax}$.

5. $u = \dfrac{y}{z}$. $y = x^x;\ z = e^x$.

6. Prove equation (12), p. 294, as suggested in the text.

Art. **10. Differentiation of implicit functions.** It often happens that we have an independent variable x and a variable y dependent on it, for which the relation is not expressed in the form

(1) $$y \equiv f(x),\ *$$

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* The symbol, $\equiv$, means "is identical with." Identities are often written with the usual sign of equality, as we have done hitherto; but the use of the symbol, $\equiv$, permits us to emphasize by the notation (whenever, as in the present instance, we may wish to do so), the fact that we are dealing with identities and not with equations of condition.

but in the form

$$(2) \qquad \phi(x, y) \equiv 0.$$

In this case y is said to be an **implicit function** of x, and *vice versa*, while in the relation (1), y is said to be an **explicit function** of x.

We could find $\frac{dy}{dx}$ by first solving the identity (2) for y in terms of x, obtaining y as an explicit function of x, and then differentiating the result. In many cases, however, the identity (2) might be too complicated to admit of solution; in many other cases the solution may be possible, and yet not convenient to effect or simple in form. In all cases, $\frac{dy}{dx}$ may be found as follows:

Let $u \equiv \phi(x, y)$, in which we regard both x and y as functions of x; then by our previous results we have

$$\frac{du}{dx} \equiv \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

But as (2) is an identity, $u \equiv 0$, we may equate the derivatives of its members, and have

$$(3) \qquad \frac{du}{dx} \equiv 0, \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \equiv 0.$$

Hence, (4)
$$\frac{dy}{dx} \equiv -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

Introducing the value of u,

$$(5) \qquad \frac{dy}{dx} \equiv -\frac{\frac{\partial \phi(x, y)}{\partial x}}{\frac{\partial \phi(x, y)}{\partial y}}.$$

For example:

I. Let the relation connecting x and y be *

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then
$$u=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

$$\frac{\partial u}{\partial x}=\frac{2\,x}{a^2};\quad \frac{\partial u}{\partial y}=\frac{2\,y}{b^2};\quad \frac{dy}{dx}=-\frac{\dfrac{2\,x}{a^2}}{\dfrac{2\,y}{b^2}}=-\frac{b^2}{a^2}\cdot\frac{x}{y}.$$

II. Let
$$xy=C.$$

Then
$$u=xy-C=0.$$

$$\frac{\partial u}{\partial x}=y;\quad \frac{\partial u}{\partial y}=x;\quad \frac{dy}{dx}=-\frac{y}{x}.$$

Whenever it is convenient to use the given relation to simplify our result, we are of course at liberty to do so. In this case we can readily solve the given relation for y, obtaining

$$y=\frac{C}{x},$$

and differentiating
$$\frac{dy}{dx}=-\frac{C}{x^2}.$$

III. Let
$$u=x^3+y^3-axy=0.$$

$$\frac{\partial u}{\partial x}=3\,x^2-ay;\quad \frac{\partial u}{\partial y}=3\,y^2-ax;\quad \frac{dy}{dx}=-\frac{3\,x^2-ay}{3\,y^2-ax}.$$

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* It was necessary above to emphasize the fact that we were dealing with identities in order that the correctness of the step by which we deduced (3) might be clear. The need for special emphasis of the character of our equations being past, we return to the use of the ordinary sign of equality, though we still continue to deal with identities.

EXERCISES XXXI

Find $\dfrac{dy}{dx}$, y being given as an implicit function of x in the following relations:

1. $xy^2 + x^2y + 1 = 0.$
2. $a^2x^2 - b^2y^2 - a^2b^2 = 0.$
3. $x^3 - 6x^2y + 2xy^2 + 7y^3 = 0.$
4. $x^3 - y^3 - xy = 0.$
5. $(x^2 - y^2)(ax + b) = 0.$
6. $x \sin y = y \sin x.$
7. $\sin\dfrac{x}{y} = \tan\dfrac{y}{x}.$
8. $x^y = y^x.$
9. $(x^2 + y^2 - ax)^2 - b^2(x^2 + y^2) = 0.$

ART. **11. Homogeneous functions.** A function of two variables is said to be **homogeneous** of **degree** n, if the function which results from multiplying each variable in it by the same constant factor c is c^n times the original function. In symbols, $f(x, y)$ is homogeneous of the nth degree in x and y, if

$$(1) \qquad f(cx, cy) = c^n f(x, y).$$

A similar definition applies to functions of any number of variables. Thus, $f(x, y, z, w)$ is **homogeneous** of the n**th degree** in x, y, z, w, if

$$(2) \qquad f(cx, cy, cz, cw) = c^n f(x, y, z, w).$$

EXAMPLES

1. $ax + by$ is homogeneous of the first degree in x and y.
2. $\dfrac{5x - 4y}{3x^2 + 9xy}$ is a homogeneous function of degree -1, in x and y.
3. $\sin\left(\dfrac{x^2 - y^2}{z^2 + w^2}\right)$ is a homogeneous function of degree zero, in x, y, z, w.
4. $\sqrt[3]{x^2} + \sqrt[3]{yz}$ is homogeneous of degree $\frac{2}{3}$ in x, y, z.
5. $\sin\dfrac{x^2 + y^2}{y^3 - x^3}$ is not homogeneous, since

$$\sin\frac{(cx)^2 + (cy)^2}{(cy)^3 - (cx)^3} = \sin\frac{x^2 + y^2}{c(y^3 - x^3)}$$

is not equal to the original function multiplied by some power of c.

EXERCISES XXXII

Determine whether or not the following functions are homogeneous; and, if homogeneous, of what degree?

1. $3x^3y - 5x^2y^2 + 7xy^3 + 6y^4.$

2. $12x^3yz^2 - 3x^4y^2 + z^6 + 5x^5z.$

3. $5x^7 + 2y^7 - 3x^5y^2 + 1.$

4. $\dfrac{3x^7y + 11x^3yz^4}{5y^6z^2 - 2x^8 + 4z^8}.$

5. $\dfrac{5x + 2y + x^2}{2x - 3y - y^2}.$

6. $\dfrac{5x}{2y^3 - 3z^3}.$

7. $\log\dfrac{x}{y}.$

8. $e^{\frac{x^2+y^2}{x^2}}.$

9. $\log\sin\dfrac{x^3}{x^2y + y^3}.$

10. $\tan\dfrac{x + y}{x - y}.$

11. $\arcsin\dfrac{x^2 + y^2}{x^4 - 3y^4}.$

12. $\dfrac{1}{\sqrt[3]{x^2 - y^2}}.$

13. $\sqrt{x^3} + y\sqrt{z} + \dfrac{z^2}{\sqrt{x}}.$

14. $\sqrt{2x^5} + \sqrt{2x^4y} - \sqrt{2x^2y^3}.$

ART. **12. Euler's theorem of homogeneous functions.** The partial derivatives of a homogeneous function possess a remarkable property discovered by Euler,* *viz.: The sum of the products formed by multiplying each partial derivative of a homogeneous function by the variable with respect to which the derivative is taken, is equal to the original function multiplied by its degree.*

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* Leonhard Euler (1707–1783) was the son of a clergyman who was himself a pupil of James Bernoulli (p. 147). Leonhard Euler was a pupil of John Bernoulli, and Nicholas the second and Daniel Bernoulli were among his fellow students. Having attained his master's degree at the age of sixteen, Euler was soon called to membership in the Academy at St. Petersburg (founded 1724). He accepted the invitation in 1727, and resigned in 1741 to become a member of the Berlin Academy, then flourishing under the fostering care of Frederick the Great. He returned to St. Petersburg in 1766, and though he became blind in the same year, he continued his scientific activity up to the time of his death. Euler was a mathematician of the first rank, and contributed largely to the development of the calculus in the eighteenth century.

We give the proof for a homogeneous function of two variables, but the method may be used equally well for any number of variables.

Let $$u = f(x, y)$$

be a homogeneous function of degree n in x and y. Then Euler's Theorem asserts that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \tag{1}$$

By the definition of homogeneous functions,

$$f(cx, cy) = c^n f(x, y). \tag{2}$$

Differentiating this identity with respect to c, we have (p. 291),

$$x\frac{\partial}{\partial x}f(cx, cy) + y\frac{\partial}{\partial x}f(cx, cy) = nc^{n-1}f(x, y). \tag{3}$$

Putting $c = 1$ in the identity (3), we have

$$x\frac{\partial}{\partial x}f(x, y) + y\frac{\partial}{\partial x}f(x, y) = nf(x, y). \tag{4}$$

or,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu. \tag{5}$$

EXERCISES XXXIII

Verify Euler's Theorem of homogeneous functions for the following functions:

1. $5x^2y^3 - 2x^5$.
2. $\dfrac{x^3}{y} - xy + \dfrac{x^2y^2}{x^2 + y^2}$.
3. $\sqrt{x} + \dfrac{2y}{\sqrt{x}} + \dfrac{xy^2}{\sqrt{x^5 - 5y^5}}$.
4. $e^{1 - \frac{x^2}{xy + y^2}}$.
5. $\arctan \dfrac{x - y}{x + y}$.
6. $\log \dfrac{x^4 + 2x^2y^2}{y^4 - 3x^3y}$.
7. $c_1x^{a_1}y^{b_1} + c_2x^{a_2}y^{b_2} + c_3x^{a_3}y^{b_3} + c_4x^{a_4}y^{b_4}$, where
$$a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = a_4 + b_4 = n.$$

8. Prove Euler's Theorem for homogeneous functions of three variables.

Verify Euler's Theorem for the following functions:

9. $ax^3y^2z + bxyz^4 + cy^6 + dxz^5$.

10. $\sin \dfrac{x+y}{z}$.

11. $\cot\sqrt{\dfrac{x+y}{x+z}}$.

12. $\sin \dfrac{x}{y} + \log \dfrac{x^2+y^2}{3\,xy - z^2}$.

ART. **13. The focal properties of the parabola.** If the equation of the parabola be written in the form (p. 21)

(1) $$F(x, y) = y^2 - 2\,px = 0,$$

we have

(2) $$\frac{\partial F}{\partial x} = -2\,p, \quad \frac{\partial F}{\partial y} = 2\,y;$$

whence

(3) $$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = \frac{2\,p}{2\,y} = \frac{p}{y}.$$

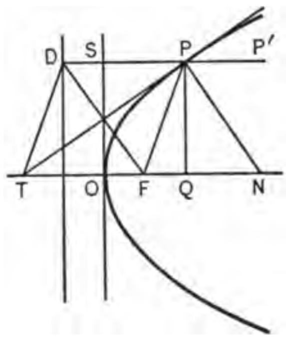

FIG. 58.

The equation of the tangent at the point P (Fig. 58), whose coördinates are x_1 and y_1, must be of the form (p. 32)

(4) $$y - y_1 = (x - x_1)\tan\tau.$$

Substituting for $\tan\tau$ its value from equation (3), and clearing of fractions, we have

$$yy_1 - y_1^2 = px - px_1;$$

and finally by (1) the equation of the tangent to the parabola becomes

(5) $$yy_1 = p(x + x_1).$$

It is apparent from (3) that the tangent at O is perpendicular to the x-axis. The point O is called the **vertex** of the parabola.

The point of intersection with the x-axis of the tangent to the parabola at the point P is found by putting $y = 0$ in (5); then

$$x + x_1 = 0, \text{ or } x = -x_1. \tag{6}$$

If T be this point of intersection, and OQ the abscissa, x_1, of P, equation (6) shows that $OT = OQ$.

This furnishes a very simple method of constructing geometrically the tangent at any point of a parabola. We drop a perpendicular from the point upon the x-axis, determine a point equally distant from the vertex with the foot of this perpendicular, but on the opposite side; the straight line through the point thus determined and the given point P is the tangent desired.

If from P we let fall a perpendicular upon the directrix of the parabola, and draw PF and DT, the figure $PDTF$ is easily shown to be a rhombus; for

$$PD = PS + SD = x_1 + \frac{p}{2},$$

$$TF = TO + OF = x_1 + \frac{p}{2}.$$

Hence $PD = TF$, and since PD and TF are parallel, and since by the definition of a parabola, $PD = PF$, the figure is a rhombus. The diagonal TP bisects the angle DPF. If at P we erect a perpendicular PN to the tangent PT, then PN forms equal angles with PF and PP'. PN is called the **normal** to the parabola at the point P. If we call PF the **focal ray** of the point P, we may formulate the theorem: *The tangent and the normal of any point of a*

parabola bisect the angles formed by the focal ray and lines parallel to the x-axis.

This property of a parabola is important in optics. For when rays of light parallel to the principal axis of a parabola impinge upon the latter, and are reflected from it, they all arrive at a common point F; for instance, PF is the reflected ray of the impinging ray PP'. This common point we have already called the **focus**. (Lat. *focus* = fireplace.) Conversely, if a source of light be at F, all rays coming from it are reflected so as to be parallel to the principal axis of the parabola. This is also true when the parabola is rotated around its principal axis so that the reflecting line becomes a reflecting surface. For this reason, concave mirrors which are intended to reflect light to a great distance should be given the parabolic form. Hertz, in the first of his celebrated experiments on the propagation of electric rays, made use of this property of parabolic surfaces. He employed large reflectors of sheet zinc bent into the form of parabolic cylinders, in whose focal line the transmitter and the receiver of the electric waves was placed. The electric rays passed from the transmitter to the first parabolic reflector were there reflected so as to become parallel, and were then reflected from the second reflector to the receiver placed at its focus. Also Marconi, in his experiments on wireless telegraphy, is attempting to confine the propagation of the electric waves to one direction by the use of copper parabolic reflectors.

Since the diagonals of the rhombus $PDTF$ bisect each other, and are perpendicular to each other, it follows that the foot of the perpendicular from the focus to any tangent of the parabola is the point of intersection of that tangent with the tangent at the vertex.

ART. **14. The focal properties of the ellipse.** If the equation of the ellipse be considered in the form

(1) $$F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

then

(2) $$\frac{\partial F}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2},$$

and therefore,

(3) $$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

The equation of the tangent at the point P, (x_1y_1) (Fig. 59), is

$$y - y_1 = (x - x_1)\tan\tau,$$

and substituting for $\tan\tau$ its value from equation (3),

$$(4)\qquad y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1),$$

or

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{{x_1}^2}{a_2} + \frac{{y_1}^2}{b^2},$$

which by equation (1) becomes

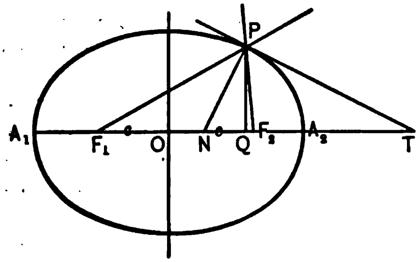

FIG. 59.

$$(5)\qquad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

The point of intersection, T, of the tangent with the x-axis is obtained by putting $y = 0$, when

$$(6)\qquad x = \frac{a^2}{x_1}.$$

This equation gives a simple geometric construction for the tangent to an ellipse. We notice first of all that the semi-minor axis b does not occur at all in the value of x, which is dependent alone upon the semi-major axis a and the abscissa x_1 of the point of tangency. If upon the same major axis $2a$ we construct various ellipses, and if P_1, P_2, P_3, P_4, etc., denote the points of these various ellipses having the same abscissa $x_1 = OQ$, then the formula shows that all the tangents at the various points P_1, P_2, P_3, $P_4\cdots$ intersect the x-axis in the same point T. Among the boundless number of ellipses which may be constructed upon $2a$ as major axis, the simplest is that in which $b = a$; *i.e.*, the circle which we have called (p. 51) the auxiliary circle of any of the ellipses upon its diameter as major axis. The point T may be determined by constructing the tangent to the circle at the point P with abscissa OQ. The lines drawn from T to the various points P_1, P_2, P_3, $P_4\cdots$ will then be tangent to the various ellipses.

From this equation the following important property may be deduced. We have already found (p. 52) that

$$PF_1 = r_1 = a + \frac{c}{a}x_1,$$

$$PF_2 = r_2 = a - \frac{c}{a}x_1;$$

hence

$$(7) \qquad \frac{PF_1}{PF_2} = \frac{a + \frac{c}{a}x_1}{a - \frac{c}{a}x_1} = \frac{a^2 + cx_1}{a^2 - cx_1}.$$

But by equation (6)

$$F_1T = OT + F_1O = \frac{a^2}{x_1} + c,$$

and

$$F_2T = \frac{a^2}{x_1} - c,$$

whence

$$(8) \qquad \frac{F_1T}{F_2T} = \frac{\frac{a^2}{x_1} + c}{\frac{a^2}{x_1} - c} = \frac{a^2 + cx_1}{a^2 - cx_1}.$$

From (7) and (8),

$$(9) \qquad F_1T : F_2T = F_1P : F_2P.$$

Further, according to a theorem of elementary geometry, the bisector of an exterior angle of a triangle divides the opposite side into segments proportional to the other two sides, and conversely, and therefore, in the triangle F_1PF_2 the straight line PT bisects the exterior angle. Calling the line PN which is perpendicular to the tangent, the **normal** to the ellipse at the point P, and calling $PF_1 = r_1$ and $PF_2 = r_2$ the **focal rays** from the point P, then it follows

easily that the normal bisects the angle made by the focal rays; *i.e.*,

At every point of an ellipse the tangent and the normal bisect, respectively, the exterior and the interior angle formed by the focal rays.

If we imagine a source of light to be placed at F_1, then PF_1 represents the path of a ray of light emanating from that point-source. If the ellipse be a reflecting line, the reflected ray must make the same angle with the normal as the incident ray, and since

$$F_1PN = NPF_2,$$

PF_2 is the reflected ray; *i.e.*,

All rays of light having as source one of the foci of an ellipse are reflected so as to meet in the other focus.

This is true of all rays which travel in straight lines and which are reflected according to the law that the angle of reflection is equal to the angle of incidence. This property of the ellipse is the cause of the peculiar echo and concentration of sound in certain arched halls, grottos, etc. It may be also illustrated by the well-known experiment of placing an easily inflammable substance in one focus of an elliptic surface and igniting it by means of a glowing coal placed in the other.

ART. 15. **The asymptotes of the hyperbola.** Writing the equation of the hyperbola in the form

$$F(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \tag{1}$$

we have

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial F}{\partial y} = -\frac{2y}{b^2}, \tag{2}$$

and

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}. \tag{3}$$

We find the equation of the tangent at the point (x_1, y_1) to be

$$y - y_1 = (x - x_1)\frac{dy}{dx} = (x - x_1)\frac{b^2x_1}{a^2y_1},$$

or
$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2};$$

whence from (1),

(4)
$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

The abscissa of the point of intersection of the tangent with the x-axis has the value

(5)
$$x = \frac{a^2}{x_1}.$$

Of particular interest are the **asymptotes,** already illustrated (p. 59) and now *defined* as the limiting positions, which the tangents approach as the points of contact move out on the hyperbola beyond all bounds. Because of the symmetry of the hyperbola it is necessary to consider it in one quadrant only, the first, for instance; as the point of contact moves out beyond all bounds, its abscissa likewise increases beyond all bounds, and we have

$$x = \lim_{x_1 \doteq \infty} \frac{a^2}{x_1}, \text{ or } x = 0;$$

that is to say, the asymptote passes through the origin. To find from equation (3) the angle which the tangent makes with the x-axis, we must ascertain the limit of the ratio of x to y when x grows beyond all bounds. By equation (1),

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}.$$

Extracting the square root, and taking the limit,

$$(6) \qquad \lim_{x \doteq \infty} \frac{y}{x} = \lim_{x \doteq \infty} \left(\frac{b^2}{a^2} - \frac{b^2}{x^2} \right)^{\frac{1}{2}} = \left(\frac{b^2}{a^2} \right)^{\frac{1}{2}} = \pm \frac{b}{a}.$$

If we substitute this value in equation (3) and denote the angle in question by ϕ, we have

$$(7) \qquad \tan \phi = \pm \frac{a}{b} \cdot \frac{b^2}{a^2} = \pm \frac{b}{a}.$$

As we are restricting our considerations to the portion of the hyperbola in the first quadrant, the positive sign is to be taken.

If we now construct a rectangle with its sides parallel to the axes, and cutting off on the axes the distances (Fig. 26, p. 55),

$$OA_1 = OA_2 = a$$

and

$$OB_1 = OB_2 = b,$$

respectively, the diagonal in the first quadrant is the required asymptote. Because of the symmetry of the hyperbola it follows that this straight line produced is the asymptote to that part of the hyperbola which lies in the third quadrant, and that the other diagonal of the rectangle is likewise an asymptote to the hyperbola in the second and in the fourth quadrant.

We have thus reached the same results for *any* hyperbola which we have previously found for the equilateral hyperbola (p. 61).

Observing that the second diagonal makes the angle $\pi - \phi$ with the x-axis, and $\tan \phi = - \tan (\pi - \phi)$, we can write the equations of the asymptotes at once from the *slopes*

(Eq. 7), and from the fact that the asymptote passes through the origin, *viz.*

$$y = \frac{b}{a}x \text{ or } \frac{y}{b} - \frac{x}{a} = 0,$$

and

$$y = -\frac{b}{a}x \text{ or } \frac{y}{b} + \frac{x}{a} = 0.$$

CHAPTER X

INFINITE SERIES

ART. 1. **Definition.** A sequence of terms which are formed according to some rule or law, so that more terms can be written according to the same rule or law, is called a series of terms, or a **series**.

For example,

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \cdots,$$

is a series, the law of which is that each term is greater by unity than the one preceding it. We could extend the series by the terms 7, 8, 9, 10, ⋯, as far as we like.

The following are further examples of series:

1. $1, \quad 3, \quad 9, \quad 27, \quad \cdots,$
2. $1, \quad 4, \quad 9, \quad 16, \quad \cdots,$
3. $1, \quad \frac{1}{5}, \quad \frac{1}{25}, \quad \frac{1}{125}, \quad \cdots,$
4. $1, \quad 3, \quad 5, \quad 7, \quad 9, \quad \cdots,$
5. $1, \quad 4, \quad 5, \quad 8, \quad 9, \quad 12, \quad 13, \quad 16, \quad 17, \quad \cdots,$
6. $10, \quad 8, \quad 6, \quad 4, \quad \cdots,$
7. $1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad \cdots.$

Exercise. Discover the law of each series by inspection, and write the next four terms of each.

If the law of the series is such that there is no bound to the number of terms which may be written, it is called an **infinite series.** A sequence of terms is not called a *series* when no

law can be discovered according to which additional terms can be written.

ART. 2. **The sum of infinite series.** The fraction $\frac{1}{3}$ may be converted into the decimal fraction 0.3333 ⋯; *i.e.* into

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000} + \cdots.$$

We say, ordinarily, that this decimal which never terminates (or the unbounded series of terms to which it is equivalent) is equal to $\frac{1}{3}$. This is not strictly true, but rather, $\frac{1}{3}$ is the limit which the decimal (or the series of terms) approaches as it is carried out farther and farther.

For, $\frac{3}{10}$ is not $\frac{1}{3}$; $\frac{3}{10} + \frac{3}{100}$, or $\frac{33}{100}$, is not $\frac{1}{3}$, but it differs less from $\frac{1}{3}$ than $\frac{3}{10}$ does; $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$, or $\frac{333}{1000}$, is not $\frac{1}{3}$, but it differs less from $\frac{1}{3}$ than either $\frac{3}{10}$ or $\frac{33}{100}$; and thus by taking more terms we may reach a sum which shall differ as little from $\frac{1}{3}$ as we like; *i.e.* the sum of the terms of the series, as we take more and more of them, approaches the limit $\frac{1}{3}$. Similarly, the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots$$

has unity as the limit of the sum of its terms.

The sum of the terms of a series as more and more are taken may not approach any limit. This is the case, for instance, in the series

$$1,\ 2,\ 3,\ 4,\ 5,\ \cdots.$$

Here evidently the sum grows large beyond all limits as more and more terms are included in it.

If the sum of the first terms of a series approaches some definite limit as more and more terms are taken, that limit is defined to be the **sum of the series.**

We often write, accordingly,

$$\tfrac{1}{3} = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots,$$
$$1 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots;$$

but these are simply abbreviations for the following:

$$\tfrac{1}{3} = \lim \{\tfrac{3}{10} + \tfrac{3}{100} + \cdots\},$$
$$1 = \lim \{\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{2} + \cdots\}.$$

A series which has a sum is called **convergent.** If the sum of the first terms of a series can be made as large as we please by taking enough terms, the series is **divergent.**

We have given one example of a divergent series above, we now add another, *viz.*,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots.$$

At first glance it may seem as if this series should be convergent, but it may be proved to be divergent as follows:

We compare the two series

$$(1)\quad 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \cdots + \tfrac{1}{16} + \cdots,$$
$$(2)\quad \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots + \tfrac{1}{16} + \cdots,$$

of which the first is the given series, and the second has every one of its terms equal to or less than the corresponding term of the first. The sum of any number of terms of (1) will therefore be greater than the sum of the same number of terms of (2), and if we can show that a sum, large at will, can be obtained by taking enough of the terms of (2), we shall thereby have shown that the same can be done by taking enough of the terms of (1).

If we write (2) thus,

$$(3)\quad \tfrac{1}{2} + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + (\tfrac{1}{16} + \cdots + \tfrac{1}{16}) + \cdots,$$

and add the terms within the parentheses, we have (2) in the form

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots.$$

By taking enough terms of this we clearly can obtain a sum large at will, and hence (1) is divergent; it is called the **harmonic series.**

The series

(4) $$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots,$$

which differs from the harmonic series only in having its signs alternately plus and minus, may be put into the two forms

(5) $$(1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \cdots,$$

(6) and $$1 - (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{4} - \tfrac{1}{5}) - (\tfrac{1}{6} - \tfrac{1}{7}) \cdots,$$

in each of which the quantities in parentheses are positive. It is easily seen that in the first form the sum of any number of terms of the series is greater than the first term $(1 - \tfrac{1}{2})$, and in the second that it is less than 1; it lies, therefore, between 1 and $\tfrac{1}{2}$.

ART. 3. **The geometric series.** The simplest example of a convergent series is the geometric series

(1) $$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \cdots,$$

for which we have the equation *

(2) $$1 + \alpha + \alpha^2 \cdots + \alpha^{n-1} = \frac{1 - \alpha^n}{1 - \alpha} = \frac{1}{1 - \alpha} - \frac{\alpha^n}{1 - \alpha}.$$

If we suppose α to be a proper fraction and n to increase without bound, then the first fraction of the right-hand

* Formula 52, Appendix.

member remains unchanged, while the second approaches zero. We obtain, therefore,

$$(3) \qquad \lim \{1 + \alpha + \alpha^2 + \alpha^3 + \cdots\} = \frac{1}{1-\alpha}.$$

The unlimited geometric series (1), *in which α is less than* 1, *is convergent, and its sum is* $\frac{1}{1-\alpha}$.

The practical applicability of infinite series depends upon the rapidity of their convergence; or, in other words, upon how many terms we must take in order to obtain a sum which shall differ as little as we wish from the limit. The most favorable case occurs when even two or three terms give a very close approximation to the limit. Taking a simple numerical example, suppose we wish to find the value in decimal notation of the fraction $\frac{0.432}{0.998}$. We put

$$\begin{aligned}\frac{0.432}{0.998} &= 0.432\frac{1}{1-0.002}\\ &= 0.432\{1 + 0.002 + (0.002)^2 + \cdots\},\end{aligned}$$

and since $0.002^2 = 0.000004$, the expression in which only the first two terms of the series are used gives correctly the first five decimal places of the value sought. If a greater accuracy be required, as to the seventh or eighth decimal place, three terms of the series suffice; for

$$\begin{aligned}\frac{0.432}{0.998} &= 0.432(1 + 0.002 + 0.000004)\\ &= 0.432 \cdot 1.002004;\end{aligned}$$

the third term is used as a correction, and each following term, if used, has a similar effect.

ART. 4. **General theorems on the convergence of series. Series with alternating signs.** Before any use is made of a

series occurring in a mathematical investigation, it must first of all be decided whether or not the series is convergent. This is often a problem of great difficulty. We can consider only the simplest cases.

From now on we shall designate the terms of any given series by

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \cdots,$$

the subscript indicating the position of a term in the series, a_m, for instance, being the mth term.

We denote the sum of the first n terms of the series by S_n. Accordingly,

$$\begin{aligned}
S_1 &= a_1,\\
S_2 &= a_1 + a_2,\\
S_3 &= a_1 + a_2 + a_3,\\
&\cdots\cdots\cdots\cdots\\
S_k &= a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k,\\
&\cdots\cdots\cdots\cdots\\
S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.
\end{aligned}$$

With this notation, and denoting the sum of a convergent series by S, the definition of the sum becomes

$$S = \lim_{n \doteq \infty} S_n.$$

If a_r is the rth term of our series, we need to consider, in order to determine whether the series converges, only the terms from a_r on; that is, the terms

(1) $$a_r + a_{r+1} + a_{r+2} + \cdots;$$

since the sum of the original series is equal to the sum of (1) plus the sum of the other $r - 1$ terms, $a_1 + a_2 + \cdots + a_{r-1}$.

A series with alternating signs can be represented by

$$(2) \qquad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots,$$

wherein $a_1, a_2, a_3, \cdots$, are positive numbers. We shall now prove the following theorem:

If the terms of an infinite series with alternating signs continually decrease numerically and approach the limit zero, the series is convergent.

The proof is similar to that given in the example (p. 312); we write the series in the forms:

$$(3) \qquad (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots,$$

$$(4) \qquad a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots.$$

The differences in the parentheses are all positive, inasmuch as we assumed the terms to decrease continually. It follows, then, from the first form, that the sum of any number of terms of the series is greater than $a_1 - a_2$, and grows larger as more terms are taken; and from the second form, that it is smaller than a_1, and grows smaller as more terms are taken. Therefore, the sum of any number of terms of the series (3) always increases as more terms are taken, and is always greater than $a_1 - a_2$, but less than a_1.

It is readily seen, further, that *if a variable quantity continually increases but always remains less than some constant, a, the variable approaches some limit.* For, either the variable may be made to differ little at will from a, in which case a is the limit, or the difference between the variable and a must always be at least equal to a certain quantity, d, say. In this case the variable, though always increasing, can never exceed $a - d$. If it approaches near at will to $a - d$, the

latter is the limit. If not, the same reasoning can be applied to $a - d$ which we have just applied to a. By thus repeatedly diminishing the quantity which the ever increasing variable can never exceed, we see that there must exist a value which the variable can never exceed, but still to which it may approach near at will. That is, the variable, while ever increasing, approaches a limit.

Applying this to the series (3), we see that the sum of more and more terms approaches a limit (lying between $a_1 - a_2$ and a_1), *i.e.* the series (3) is convergent.

Similarly, the series (4) may be shown to approach a limit lying between $a_1 - a_2$ and a_1.

We show, finally, that (3) and (4) approach the *same* limit. The nth term of (3) is $a_{2n-1} - a_{2n}$, and the nth term of (4) is $-(a_{2n} - a_{2n+1})$. The difference between the two sums of n terms from each series is a_{2n+1}, and this term by hypothesis approaches zero as n grows beyond all bounds. Calling the sums S_n and S'_n, we may write

$$S_n = S'_n + \epsilon,$$

and taking the limit,

$$\lim S_n = \lim S'_n.$$

EXERCISES XXXIV

In the following series, first discover the law by inspection, then write several terms more, then the nth term; decide whether or not the series is convergent, and if so, between what values the limit must lie.

1. $8 - 4 + 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots.$
2. $a - a^2 + a^3 - a^4 + a^5 - a^6 + \cdots.$ $(a < 1)$.
3. $2 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots.$
4. $1 - \frac{1}{a} + \frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} - \frac{1}{a+4} + \cdots.$

5. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{13}+\frac{1}{35}-\cdots.$

6. $a-\dfrac{1}{a+k}+\dfrac{1}{a^3+2k}-\dfrac{1}{a^5+3k}+\dfrac{1}{a^7+4k}-\cdots.$

7. $a-\dfrac{a^2}{2}+\dfrac{a^3}{3}-\dfrac{a^4}{4}+\dfrac{a^5}{5}-\dfrac{a^6}{6}+\cdots.$ $(a<1).$

8. $\dfrac{a}{a+1}-\dfrac{a+1}{a+2}+\dfrac{a+2}{a+3}-\dfrac{a+3}{a+4}+\cdots.$

9. $\dfrac{a+1}{a}-\dfrac{a+2}{a+1}+\dfrac{a+3}{a+2}-\dfrac{a+4}{a+3}+\cdots.$

10. $(a+x)-(a^2+x^2)+(a^3+x^3)-(a^4+x^4)+\cdots.$ $(a<1;\ x<1).$

11. $2-\frac{4}{3}+\frac{10}{9}-\frac{28}{27}+\frac{82}{81}-\frac{244}{243}+\cdots.$

12. $\dfrac{1}{1\cdot 2}+\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 6}+\dfrac{1}{7\cdot 8}+\cdots.$

HINT: $\dfrac{1}{5\cdot 6}=\dfrac{1}{5}-\dfrac{1}{6}$, etc.

13. $a-\dfrac{a^2}{2}+\dfrac{a^3}{3}-\dfrac{a^4}{4}+\cdots.$

14. $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots.$

15. $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots.$

ART. 5. **Series with varying signs.** In case there is only a limited number of terms having one of the signs (for instance, if there are only r negative terms), then there will be a last one of these terms, and from this term on all the terms will be of like sign, and the convergency will be determined under the rules for series all of whose terms are of like sign. (In the instance given, if a_n be the rth (and last) negative term, the series will be convergent or not, according as $a_{n+1}+a_{n+2}+\cdots$, with all positive terms, is convergent or not.)

In case there is a boundless number of terms of either sign, the following theorem will enable us, in this case also, frequently to determine the convergency by means of the theorems for series of like sign.

A series with varying signs is convergent if the series deduced from it by making all signs positive is convergent.

By hypothesis, there is a boundless number both of positive and of negative terms, and the series resulting from making all signs positive is convergent. We now consider two infinite series, one made up of the positive terms of the given series, the other of its negative terms taken positively. Both of these series must be convergent, since each is either convergent or grows beyond all bounds; and if the latter, the original series would also increase beyond all bounds when its terms are all taken positively.

Let L_1 and L_2 be the limits of these series, and T_n and U_n the sums of their first n terms respectively, and let S_n denote, as usual, the sum of the first n terms of the given series. Then

$$S_n = T_q - U_r, \quad (q + r = n),$$

and both q and r will grow large without bound as n does so.

But $$T_q = L_1 + \epsilon_q,$$
and $$U_r = L_2 + \epsilon_r,$$

where ϵ_q and ϵ_r approach zero as q and r increase without bound.

Hence, $$S_n = L_1 - L_2 + \epsilon_q - \epsilon_r,$$
and $$\lim_{n \doteq \infty} S_n = L_1 - L_2,$$

or the given series is convergent.

The theorem just proved states a condition which is sufficient but not necessary for the convergence of series with varying signs. For instance, we have proved (p. 313) that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

is convergent, and (p. 312) that the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots$$

is divergent.

ART. 6. **Series whose signs are all positive.** We now turn to series all of whose terms are of like signs, which, without loss of generality, we may assume to be positive. We may represent such a series by

$$(1) \qquad a_1 + a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1} + \cdots,$$

where we consider all the quantities $a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots$, to be positive.

In this case, the more terms we take the larger their sum will become. To prove the series convergent, it is necessary and sufficient to show that no matter how many terms we take, their sum cannot be larger than some definite quantity (p. 316).

This may often be done by the following theorem:

If in an infinite series of positive terms only, the quotient of every term divided by that which precedes it is, from a certain term on, less than some quantity which is itself less than unity, the series is convergent.

To prove this theorem, we assume that all the quotients

$$\frac{a_{r+1}}{a_r}, \quad \frac{a_{r+2}}{a_{r+1}}, \quad \frac{a_{r+3}}{a_{r+2}}, \quad \cdots$$

are less than some quantity q which is itself less than unity. We have then the following relations:

$$\frac{a_{r+1}}{a_r} < q, \quad \frac{a_{r+2}}{a_{r+1}} < q, \quad \frac{a_{r+3}}{a_{r+2}} < q, \quad \frac{a_{r+4}}{a_{r+3}} < q,$$

etc.

Taking the first relation to start with, and then the products, first of the first two, then of the first three, then of the first four, etc., etc., we find

$$\frac{a_{r+1}}{a_r} < q, \quad \text{or} \quad a_{r+1} < q \cdot a_r\,;$$

$$\frac{a_{r+2}}{a_r} < q^2, \quad \text{or} \quad a_{r+2} < q^2 \cdot a_r\,;$$

$$\frac{a_{r+3}}{a_r} < q^3, \quad \text{or} \quad a_{r+3} < q^3 \cdot a_r\,;$$

$$\frac{a_{r+4}}{a_r} < q^4, \quad \text{or} \quad a_{r+4} < q^4 \cdot a_r\,;$$

$$\cdots\cdots\cdots\cdots$$

and by adding

(2) $$a_{r+1} + a_{r+2} + a_{r+3} + \cdots < qa_r + q^2a_r + q^3a_r + \cdots\,;$$

and, on adding a_r to each member,

(3) $$a_r + a_{r+1} + a_{r+2} + \cdots < a_r(1 + q + q^2 + q^3 + \cdots).$$

But by equation (3) (p. 314),

$$\lim\,[1 + q + q^2 + \cdots] = \frac{1}{1-q};$$

hence

(4) $$a_r + a_{r+1} + a_{r+2} + \cdots < \frac{a_r}{1-q}.$$

Accordingly, the series (3), and therefore the series (1), is convergent.

Exercise. Prove similarly that if in any series, whose terms are all positive, the ratio of each term to the preceding is, from a certain term on, greater than or equal to unity, this series is divergent.

We mention here two theorems easily seen to be true, which may often be applied with advantage. If in any given series,

(5) $$a_1 + a_2 + a_3 + \cdots,$$

all of its terms after a certain one are smaller than the corresponding terms of a series known to be convergent, then the series (5) is also convergent.

If, however, all the terms of (5), from a certain one on, are greater than the corresponding terms of a series known to be divergent, then the series (5) is divergent.

The theorems which we have proved will usually suffice to determine the question of convergency in the simple series which we shall take up. The discussion of the ratio $\frac{a_{m+1}}{a_m}$ is especially effective.

ART. 7. **Rapidity of convergency.** We have already stated (p. 314) that the practical applicability of infinite series depends upon their rapidity of convergency; the greater the number of the terms which must be considered, the less, of course, is the application of the series to be recommended. To ascertain how many terms should be taken to secure any desired degree of approximation to the limit of the series, it is necessary to be able to determine how great the error will be if the series is discontinued at any given term; this may readily be done in the cases of the series we have hitherto treated. The error in the geometric series is given at once by equation (2), p. 313. Writing that equation in the form

$$(1) \qquad \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha},$$

we see that the value obtained by taking the first n terms will differ by $\frac{\alpha^n}{1-\alpha}$ from the limit of the series. On substituting $-\beta$ for α, we obtain

$$(2) \quad \frac{1}{1+\beta} = 1 - \beta + \beta^2 - \beta^3 + \cdots + (-1)^{n-1}\beta^{n-1} + (-1)^n \frac{\beta^n}{1+\beta}.$$

It now follows that the errors made by taking the first n terms of the series in question, instead of the limits

$$\frac{1}{1-\alpha} \text{ and } \frac{1}{1+\beta},$$

are

$$\frac{\alpha^n}{1-\alpha} \text{ and } (-1)^n\frac{\beta^n}{1+\beta},$$

respectively; (in the case of the second series, the sum is alternately too large and too small).

For the series (p. 320)

$$a_1 + a_2 + a_3 + \cdots$$

it follows, similarly, from equation (4), p. 321, that the error occasioned by putting for this series the series

$$a_1 + a_2 + \cdots + a_{r-1}$$

is less than

(3) $$\frac{a_r}{1-q},$$

where q has been defined (p. 320).

ART. 8. **Application to the series for e.** We found (p. 138)

$$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots.$$

Two questions arise: I. Is this series convergent? II. What is the error committed by discontinuing the series at any given term?

Forming the ratio of two consecutive terms, a_m and a_{m+1}, we have for this series,

$$\frac{a_{m+1}}{a_m} = \frac{1}{m}.$$

Since, from the third term on, $\frac{1}{m}$ is less than one-half, the series is convergent according to p. 320. If we express its approximate value by the sum of the first m terms and put $q = \frac{1}{m}$, the error committed is less than

$$\frac{a_{m+1}}{1-\frac{1}{m}} = a_{m+1} \cdot \frac{m}{m-1} = \frac{1}{m!}\left(1 + \frac{1}{m-1}\right).$$

In particular, a value obtained by taking only the first seven terms into consideration, differs from the true value by less than

$$\frac{1}{7!}\left(1 + \frac{1}{6}\right) = \frac{1}{4320} = 0.00023 \cdots,$$

a result with which the value found on p. 138 agrees.

EXERCISES XXXV

Determine whether or not each of the following series is convergent:*

1. $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \frac{7}{64} + \cdots.$
2. $1 + \frac{5}{2!} + \frac{9}{3!} + \frac{13}{4!} + \frac{17}{5!} + \cdots.$
3. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots.$
4. $1 + a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \cdots$ when $a < 1$.
5. $1 + \frac{2^2}{1} + \frac{3^3}{2^2} + \frac{4^4}{3^3} + \frac{5^5}{4^4} + \cdots.$
6. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots.$
7. $1 + \frac{1}{3} + \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \cdots.$
8. $1 + \frac{2}{2^2} + \frac{4}{3^3} + \frac{6}{4^4} + \frac{8}{5^5} + \frac{10}{6^6} + \frac{12}{7^7} + \cdots.$

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* The determination may sometimes be made in more ways than one by applying different ones of the previous criteria.

9. $1 + \frac{1}{2} + \frac{2}{3} + \frac{1}{4} + \frac{2}{5} + \frac{2}{6} + \cdots.$

10. $\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots.$

11. $1 + \frac{2a}{2} + \frac{3a}{4} + \frac{4a}{8} + \frac{5a}{16} + \frac{6a}{32} + \cdots$ when $a < 1$.

12. $1 + \frac{x}{1} + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \frac{x^4}{4^4} + \cdots$ when $x < 1$.

13. $x + 2^5x^2 + 3^5x^3 + 4^5x^4 + 5^5x^5 + 6^5x^6 + \cdots$ when $x < 1$.

14. $1 + \frac{x}{1+1} + \frac{x^2}{4+2} + \frac{x^3}{9+3} + \frac{x^4}{16+4} + \cdots$ when $x < 1$.

15. Consider the series of No. 14 when $x > 1$.

ART. 9. **Maclaurin's Theorem.** It is a common procedure in scientific investigations to try to substitute an approximate formula, obtained empirically, for a law representing the unknown course of a process of nature. Considering, for instance, the expansion of a rod that at the temperature 0° C. has unit length, the crudest supposition which we can make is that the length of the rod does not vary with the temperature. This supposition, which may be expressed by the formula

$$(1) \qquad l = 1,$$

is sufficiently accurate for many practical purposes. If we make the supposition that the expansion is proportional to the temperature, we can represent the length of the rod at the temperature θ by the formula

$$(2) \qquad l = 1 + \alpha\theta,$$

in which α is the coefficient of expansion. The formula (2) gives a closer approximation to the length at any temperature than does (1), while the following is a formula corresponding still more closely to the actual length,

$$(3) \qquad l = 1 + \alpha\theta + \beta\theta^2,$$

where α and β are constants, which may be determined by comparing the formula with the results of observation. For example, the formula for the linear expansion of a rod of platinum has been found to be

$$l = 1 + 0.00000851\,\theta + 0.0000000035\,\theta^2,$$

where θ is the temperature; it is apparent that the term $\beta\theta^2$ has the character of a correction, rendering the value of l more nearly exact; in this particular case the correction has so slight a value that the formula is accurate enough for all practical needs. If this should not happen to be the case, we could proceed a step farther and add a third term, as $\gamma\theta^3$ (a correction to the correction), etc., etc.

The question arises whether this formula could not, by the proper corrections, be made absolutely accurate. It is at once evident that to attain absolute accuracy we must take all possible corrections into account, so that the formula may become an infinite series, as

$$1 + \alpha\theta + \beta\theta^2 + \gamma\theta^3 + \cdots,$$

and it is easily seen that such series would be convergent from the very nature of the separate terms.

In the example treated above it was required to find a formula giving the length l of the rod for any value whatever of the temperature. The length l is, therefore, a function of the temperature θ, and the office of the formula is to give expression to this unknown function, $f(\theta)$. If, to treat the problem generally, we denote the variable upon which the function depends by x and the function by $f(x)$, we wish to express $f(x)$ in the form of an infinite series whose terms increase by powers of x,

$$(4) \qquad f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 \cdots,$$

where A, B, C, D, E, $\cdots$, are definite but as yet unknown numbers. As soon as the existence of such a series is established, our whole problem resolves itself into the determination of the values of A, B, C, D, E, $\cdots$.

Putting $x = 0$ in (4), then

$$f(0) = A,$$

that is, A is the value which the function assumes when $x = 0$, just as in formulæ (2) and (3) the constant term 1 of the right side gives the length of the rod for the temperature $\theta = 0$. If we now take the derivative of each member of equation (4), we have

$$(5) \qquad f'(x) = B + 2\,Cx + 3\,Dx^2 + 4\,Ex^3 + \cdots,$$

and if we now put $x = 0$, we find

$$(6) \qquad f'(0) = B; \quad B = \frac{f'(0)}{1},$$

that is, B is equal to the value which the derivative $f'(x)$ assumes when x is given the value zero. On forming the derivative of equation (5), we have

$$(7) \qquad f''(x) = 2\,C + 2 \cdot 3\,Dx + 3 \cdot 4\,Ex^2 + \cdots,$$

and, after putting $x = 0$, we obtain

$$(8) \qquad f''(0) = 2\,C; \quad C = \frac{f''(0)}{1 \cdot 2},$$

where $f''(0)$ denotes the value which $f''(x)$ assumes for $x = 0$. If we differentiate again, we find

$$(9) \qquad f'''(x) = 1 \cdot 2 \cdot 3\,D + 2 \cdot 3 \cdot 4\,Ex + \cdots,$$

that is,

$$(10) \qquad f'''(0) = 1 \cdot 2 \cdot 3\,D; \quad D = \frac{f'''(0)}{1 \cdot 2 \cdot 3},$$

etc., etc. In this way we have determined the unknown coefficients in a simple manner; and for the function in question, $f(x)$, we have the series

$$(11) \qquad f(x)=f(0)+\frac{x}{1}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots,$$

which is called **Maclaurin's Series.***

It is a series of great fruitfulness and importance, yielding expansions by whose aid the approximate value of many functions can readily be computed for any given value of the variable.

ART. **10.** **The series for e^x, sin x and cos x.** We at once apply Maclaurin's Series to several simple functions.

I. Let $f(x)=e^x.$

The successive derivatives† are

$$f'(x)=e^x,\ f''(x)=e^x,\ f'''(x)=e^x\cdots,$$

and

$$f(0)=1,\ f'(0)=1,\ f''(0)=1,\ f'''(0)=1,\ \cdots.$$

Maclaurin's Series accordingly assumes the form

$$(1) \qquad e^x=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,$$

* Colin Maclaurin (1698–1746) had even at the age of 15 discovered many of the theorems which he published later, and before he had attained the age of 20 was appointed Professor of Mathematics at Aberdeen; from 1725 to 1745 he was Professor of Mathematics at Edinburgh, and joined in the defense of the city against the "Young Pretender" in 1745; upon the capture of the city by the latter, he fled to York, where he died in 1746. The series which we have treated appeared in the *Treatise of Fluxions*, 1742, and is a special case of Taylor's Series, which we shall take up later, having been recognized as such by Maclaurin.

† In applying Maclaurin's Series, it is better to form all the derivatives needed first (simplifying as much as possible at each step), and then to put $x=0$ in each.

which is called the **exponential series.** With its aid we can compute the value of e^x for a given value of x, for $x = 1$, we have the series already deduced for e (p. 323).

II. If we put

$$f(x) = \sin x, \tag{2}$$

we have the derivatives in order as follows:

$$f'(x) = \cos x, \quad f''(x) = -\sin x,$$
$$f'''(x) = -\cos x, \quad f^{\text{iv}}(x) = \sin x, \quad f^{\text{v}}(x) = \cos x;$$

whence

$$f(0) = f''(0) = f^{\text{iv}}(0) = \cdots = 0,$$
$$f'(0) = 1, \ f'''(0) = -1, \ f^{\text{v}}(0) = 1 \cdots;$$

we therefore have

$$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots. \tag{3}$$

III. Analogously we find for

$$f(x) = \cos x, \tag{4}$$

$$f'(x) = -\sin x, \ f''(x) = -\cos x, \ f'''(x) = \sin x,$$
$$f^{\text{iv}}(x) = \cos x, \quad f^{\text{v}}(x) = -\sin x;$$

whence

$$f'(0) = f'''(0) = f^{\text{v}}(0) = \cdots = 0,$$
$$f(0) = 1, \ f''(0) = -1, \ f^{\text{iv}}(0) = 1 \cdots,$$

and we have for $\cos x$ the following series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots. \tag{5}$$

It is to be observed that, according to the conventions of p. 74, x is always to be considered as the magnitude of the angle in circular measure. If, for example, we compute

the values of sin 1 and cos 1, that is, the sine and cosine of the unit angle of circular measure, the radian (p. 74), we find

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!} + \cdots,$$

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \cdots.$$

Using the numerical values calculated on p. 138, we have

$$\begin{array}{llll}
1 = 1 & \dfrac{1}{2!} = 0.5 & \dfrac{1}{1!} = 1 & \dfrac{1}{3!} = 0.1667 \\
\dfrac{1}{4!} = 0.0417 & \dfrac{1}{6!} = \underline{0.0014} & \dfrac{1}{5!} = \underline{0.0083} & \dfrac{1}{7!} = \underline{0.0002} \\
 & \qquad 0.5014 & \qquad 1.0083 & \qquad 0.1669 \\
\dfrac{1}{8!} = \underline{0.0000} & & & \\
\qquad 1.0417 & & &
\end{array}$$

that is, we find

$$\cos 1 = 1.0417 - 0.5014 = 0.5403,$$

$$\sin 1 = 1.0083 - 0.1669 = 0.8414,$$

while the values correct to four places of decimals are 0.5403 and 0.8415, respectively. The fourth place of our result is thus correct in the first case, and in the second case the difference amounts merely to a unit in the fourth place.*

These are the results of making numerical substitutions in the series, but before we have confidence in their correctness

* In making computations of approximate values by means of infinite series, it is better to carry the decimal approximation for each term several places of decimals further than the number of places to which the result is desired to be correct, and to compute the successive terms of the series until they no longer present any significant figures. In the value thus obtained the last figure will always be untrustworthy, but unless the series is quite slowly convergent, all but the last *two* figures will be correct.

we must ascertain whether these values of x (and, in general, what values of x) make our series convergent. According to pp. 318–20, any series whatever is certainly convergent if, from a certain term on, the quotient of a_{n+1} and a_n is always less than some number which is itself less than unity. In the series for e^x the terms a_n and a_{n+1} have the values

$$a_n = \frac{x^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!};$$

consequently,

$$\frac{a_{n+1}}{a^n} = \frac{x^{n+1}}{(n+1)!} : \frac{x^n}{n!} = \frac{x}{n+1};$$

and no matter what the value of the number x, this quotient, when n increases without bound, is not only, from a certain term on, always less than some number which is itself less than unity, but indeed approaches the value zero. If, for example, $x = 100$, then when n receives the successive values 100, 101, 102, ⋯, that is, from the hundredth term on, the quotients are, respectively,

$$\tfrac{100}{101}, \tfrac{100}{102}, \tfrac{100}{103}, \cdots,$$

and the series is undoubtedly convergent. Naturally, the greater the value of x, the greater the number of terms of the series needed for the numerical computation of an approximate value of e^x.

The series (3) and (5) may likewise be proved convergent for all values of x by means of the theorems which we have established. We leave this as an exercise for the reader. We need to employ the series for sine and cosine only for angles which lie between 0 and $\frac{\pi}{4}$; for the values of the functions for all other angles can be calculated from these.*

* Formulæ 13 *et seq.*, Appendix.

In this way the values of $\sin x$ and $\cos x$ may actually be determined, and we see from the above example that but few terms are needed to obtain fairly correct results.

EXERCISES XXXVI

By means of Maclaurin's Theorem, show that

1. $\cos mx = 1 - \frac{m^2x^2}{2!} + \frac{m^4x^4}{4!} - \frac{m^6x^6}{6!} + \cdots.$

2. $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots.$

3. $\log\cos x = -\frac{x^2}{2!} - \frac{2x^4}{4!} - \cdots.$

4. $\sqrt{a+x} = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}x + \cdots.$

5. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots.$

6. $\sqrt[3]{e^x + a} = (1+a)^{\frac{1}{3}} + \frac{x}{3(1+a)^{\frac{2}{3}}} + \frac{x^2}{9(1+a)^{\frac{5}{3}}} + \cdots.$

7. $\sin^2 x = x^2 - \frac{2x^4}{3!} + \frac{32x^6}{6!} + \cdots.$

8. $\cos^2 x = 1 - x^2 + \frac{8x^4}{4!} - \frac{32x^6}{6!} + \cdots.$

   HINT: Use Formula 36, Appendix.

9. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots.$

10. $e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{4x^5}{5!} - \frac{8x^6}{6!} - \frac{8x^7}{7!} + \frac{16x^9}{9!} + \cdots.$

ART. 11. **The series for $\tan x$.** To deduce the series for $\tan x$ by the aid of Maclaurin's Theorem, we need the values which $\tan x$ and its higher derivatives assume for $x = 0$. These values cannot be so directly determined as was the case with e^x, $\sin x$, and $\cos x$, hence we make use of an artifice

which may also be employed to advantage in the development of other functions.

If we put

$$(1) \qquad y = \tan x = \frac{\sin x}{\cos x},$$

or

$$(2) \qquad y \cos x = \sin x,$$

and using the notation (p. 273),

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''' \cdots \frac{d^ny}{dx^n} = y^{(n)},$$

we find, by successive differentiation,

$$(3) \qquad y' \cos x - y \sin x = \cos x;$$

$$(4) \qquad y'' \cos x - 2\,y' \sin x - y \cos x = -\sin x;$$

$$(5) \qquad y''' \cos x - 3\,y'' \sin x - 3\,y' \cos x + y \sin x = -\cos x.$$

By continuing the process of differentiation, it soon appears that, however often we differentiate, the numerical coefficients of the left member follow the same law of formation as those of the Binomial Series,* so that after n differentiations the left member would assume the form

$$(6) \qquad \begin{aligned} y^{(n)} \cos x - \frac{n}{1} y^{(n-1)} \sin x - \frac{n(n-1)}{1 \cdot 2} y^{(n-2)} \cos x \\ + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} y^{(n-3)} \sin x + \cdots, \end{aligned}$$

while the right member is the nth derivative of $\sin x$ (p. 274). The proof can easily be made by mathematical induction.†

* Formula 3, Appendix.

† A similar relation holds for the nth derivative of the product of any two functions. The general theorem is due to Leibnitz, and is usually called **Leibnitz's Theorem.**

From these expressions we can find the values which $\tan x$ and its derivatives assume for $x = 0$. If they be denoted by

$$(y)_0, \ (y')_0, \ (y'')_0, \ (y''')_0, \cdots,$$

it follows from (2), (3), (4), etc., that

$$(y)_0 = (y'')_0 = (y^{\text{iv}})_0 = \cdots = 0,$$

$$(y')_0 = 1, \ (y''')_0 = 2, \ (y^{\text{v}})_0 = 16, \ \cdots;$$

whence

$$\tan x = \frac{x}{1} + \frac{x^3}{1 \cdot 2 \cdot 3} \cdot 2 + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 16 + \cdots \tag{7}$$

$$= \frac{x}{1} + \frac{x^3}{3} + \frac{2\,x^5}{15} + \cdots.$$

ART. 12. **Taylor's Theorem.** To determine the series for e^x, $\sin x$, and $\cos x$, we had to make use of the values which these functions and their derivatives assume when $x = 0$. Every function, however, cannot be treated in this way; for example, $\log x$ and all of its derivatives grow large without bound when $x \doteq 0$. In order to develop such functions into series, we make use of another series due to Taylor, of which Maclaurin's Series is a special case.

Just as Maclaurin's Series operates with the values which the function and its derivatives assume when $x = 0$, Taylor's Series is based upon the values which the function and its derivatives take when $x = h$, h being any given quantity. It gives the value of the function $f(x + h)$ when all the values

$$f(h), \ f'(h), \ f''(h), \ \cdots,$$

are known. Taylor's Series can be deduced in exactly the same way as Maclaurin's. We start with the assumption that we may express $f(x + h)$ in the form

$$f(x + h) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots; \tag{1}$$

putting $x+h=y$, we have

$$f(y)=A+B(y-h)+C(y-h)^2+D(y-h)^3+E(y-h)^4+\cdots.$$

Differentiating successively with respect to y, we have

$$\begin{aligned}
f'(y) &= B+2\,C(y-h)+3\,D(y-h)^2+4\,E(y-h)^3+\cdots,\\
f''(y) &= 2\,C+2\cdot 3\,D(y-h)+3\cdot 4\,E(y-h)^2+\cdots,\\
f'''(y) &= 2\cdot 3\,D+2\cdot 3\cdot 4\,E(y-h)+\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Putting $y=h$, *i.e.* $x=0$, we obtain

$$\begin{aligned}
f(h) &= A, & f''(h) &= 2\,C,\\
f'(h) &= B, & f'''(h) &= 2\cdot 3\,D,\\
&\cdots\cdots & &\cdots\cdots
\end{aligned}$$

Substituting the values of A, B, C, D, $\cdots$, thus obtained in (1), it becomes

$$(2)\qquad \boldsymbol{f(x+h)=f(h)+f'(h)x+\frac{f''(h)}{2!}x^2+\frac{f'''(h)}{3!}x^3+\cdots.}$$

This is the expansion known as **Taylor's Series.*** By interchanging x and h, it may also be written

$$(3)\qquad f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2!}h^2+\frac{f'''(x)}{3!}h^3+\cdots.$$

The error committed by discontinuing Taylor's or Maclaurin's Series after any term is equal to the sum of all the omitted terms. The value of this sum or remainder can be estimated or expressed in a formula. It would be beyond the scope of this book, however, to take up the determina-

* Published by Taylor in 1715 in his *Methodus Incrementorum.* Brook Taylor (1685–1731), Doctor of Laws, although a jurist, devoted considerable attention to mathematics, and was proficient in music and painting as well.

tion of the term equivalent to the remainder of the series after any given term. The plan of this book likewise does not permit us to take up the question of the *convergency* of Taylor's Series, or the proof that the *assumption* from which we started was correct. These and other questions remain unsettled above, and our discussions must be characterized rather as making the truth of the theorem plausible than as a rigorous proof. Strict determinations of all the questions involved have, however, been made, including the values of x for which the series converges.

To show the ease and rapidity with which a function may be developed into a series by the aid of Taylor's Theorem, we carry out the development of a function first according to the rules of algebra and then by the theorem under discussion.

Suppose we have given the function

$$(4) \qquad y = f(x) = ax^3 + bx^2 + cx + d.$$

If x receive the increment h, we have

$$(5) \qquad f(x+h) = a(x+h)^3 + b(x+h)^2 + c(x+h) + d;$$

the right-hand member of this equation when expanded becomes

$$\begin{aligned} f(x+h) = ax^3 &+ 3\,ax^2h + 3\,axh^2 + ah^3 \\ &+ \quad bx^2 \quad + 2\,bxh + bh^2 \\ &\qquad\qquad + \quad cx \quad + ch \\ &\qquad\qquad\qquad\qquad + d, \end{aligned}$$

or (6) $\quad f(x+h) = (ax^3 + bx^2 + cx + d) + (3\,ax^2 + 2\,bx + c)\,h$

$$+ (3\,ax + b)\,h^2 + ah^3.$$

If we apply Taylor's Theorem to the function in hand, *viz.:*

$$f(x) = ax^3 + bx^2 + cx + d,$$

we find $$f'(x) = 3\,ax^2 + 2\,bx + c,$$

$$f''(x) = 6\,ax + 2\,b\,; \quad \frac{f''(x)}{2!} = 3\,ax + b,$$

$$f'''(x) = 6\,a\,; \quad \frac{f'''(x)}{3!} = a,$$

$$f^{\text{iv}}(x) = 0.$$

Substituting these values in equation (2), we have

$$(7)\quad f(x+h) = (ax^3 + bx^2 + cx + d) + (3\,ax^2 + 2\,bx + c)\,h + (3\,ax + b)\,h^2 + ah^3,$$

the same series which was found above by purely algebraical methods. The series terminates, because the derivatives from the fourth on are all zero. This example shows that even simple processes of algebra may be performed more expeditiously by the use of Taylor's Series, though, of course, the chief value of the series is for the expansion of functions which are quite beyond the reach of algebra.

ART. **13. The logarithmic series.** Let

$$(1)\qquad f(x) = \log x\,;$$

then (p. 274)

$$(2)\quad f'(x) = \frac{1}{x}, \qquad f'''(x) = \frac{1 \cdot 2}{x^3},$$

$$f''(x) = -\frac{1}{x^2}, \qquad f^{\text{iv}}(x) = -\frac{1 \cdot 2 \cdot 3}{x^4}, \;\ldots,$$

We put $x = 1$, since this gives the foregoing derivatives the simplest form, and makes the computation of the series the most convenient. Then

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 1 \cdot 2,$$
$$f^{\text{iv}}(1) = -1 \cdot 2 \cdot 3,\ \cdots,$$

and on substituting these values in Taylor's formula, we have

$$\text{(3)} \qquad \log(1 + h) = \frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots.$$

This series can be employed in the determination of the logarithms of numbers greater than unity, provided the series is convergent. The logarithms of numbers less than unity are obtained by giving h in the last equation a negative value, so that

$$\text{(4)} \qquad \log(1 - h) = -\frac{h}{1} - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4} - \cdots.$$

The circumstance that all the signs are now negative agrees with the fact that the logarithms of numbers less than unity have negative values.

According to p. 322, each of the foregoing series must certainly converge as soon as $h < 1$, since every term of each of the two series is smaller than the corresponding term of a geometric series with the ratio h. If $h > 1$, the series is divergent by p. 321. Let the student make the proof in detail.

If we put $h = 1$, equation (4) gives as the value of log (0),

$$\log 0 = -\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots\}.$$

As is well known, log 0 is negative and infinitely large, and we have already convinced ourselves (p. 312) that the

series in the right member is divergent. Equation (3), on the other hand, yields, when $h = 1$, the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots,$$

which is convergent (p. 313). This series gives the value of log 2, and its sum is approximately 0.69325.

The series above cannot be used at all to compute logarithms of numbers greater than 2; even for numbers less than 2, it converges slowly.

The series which is ordinarily used in the computation of logarithms, and which is convergent for all values of the variable greater than unity, is deduced as follows:

Since $$\log \frac{1+h}{1-h} = \log(1+h) - \log(1-h),$$

we get by subtracting (4) from (3),

$$\log \frac{1+h}{1-h} = 2\left\{h + \frac{h^3}{3} + \frac{h^5}{5} + \frac{h^7}{7} + \cdots\right\}.$$

This series is convergent if $h < 1$, since it is the difference of two series, both of which are convergent if $h < 1$. If, now, N be any number greater than unity, we may put

$$N = \frac{1+h}{1-h},$$

whence $$h = \frac{N-1}{N+1},$$

which is always a proper fraction, and will therefore make the series above convergent.

Substituting this value of h in the series above, we should obtain a series for $\log N$ convergent for all values of N greater than unity. The series so obtained does not, however, converge very rapidly, and another converging much more rapidly is obtained as follows:

Put
$$\frac{n+1}{n}=\frac{1+h}{1-h},$$
then
$$h=\frac{1}{2n+1}.$$

Here again h is less than unity for all values of n greater than unity, and therefore this value of h will make the series above convergent. By substituting, we obtain

$$\log\left(\frac{n+1}{n}\right)=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\},$$
or
$$\log(n+1)=\log n+2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\}.$$

This series is rapidly convergent and enables us to compute readily the value of $\log(n+1)$ when that of $\log n$ is known. Log 1 being 0, we compute log 2, then log 3, etc.

These are logarithms to the base e, since in our differentiations we assumed that we were dealing with this base. The mode of passing from logarithms to the base e to those to any other base, 10, for instance, as well as the details of the actual computations, are explained in works on trigonometry.

ART. **14. The Binomial Theorem.** *Given*

(1) $$f(x)=x^n,$$

where n may be positive or negative, an integer or a fraction, to find $(x+h)^n$.

By differentiation

$$f'(x)=nx^{n-1},$$
$$f''(x)=n(n-1)x^{n-2},$$
$$f'''(x)=n(n-1)(n-2)x^{n-3},$$
etc., etc.

From these values,

$$\begin{aligned} f\ \ (h) &= h^n, \\ f'\ (h) &= nh^{n-1}, \\ f''\ (h) &= n(n-1)h^{n-2}, \\ f'''(h) &= n(n-1)(n-2)h^{n-3}. \\ \cdots, &\quad\ \cdots. \end{aligned}$$

Applying Taylor's Series to the expansion of $(x+h)^n$, and using these values, we have

$$(x+h)^n = h^n + nh^{n-1}x + \frac{n(n-1)}{2!}h^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}h^{n-3}x^3 + \cdots.$$

If $h = 1$, this expression becomes

(2)
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

This series is known as the **Binomial Series.** It is true for every value of n, positive or negative, integral or fractional. There is a difference in form between this series and the Binomial Theorem for positive integral exponents. The number of terms of series (2) is in general boundless; the coefficients are in general all different from zero; they can assume the value zero only when one of the numbers

$$(n-1),\ (n-2),\ (n-3),\ (n-4),\ \cdots,$$

can become zero; that is, n must be a positive integer. In that case the series terminates, and has a finite number of terms.

It can be proved without much difficulty that whatever the value of n, our series converges, if x is less than unity. The proof is left to the student as an exercise.

EXERCISES XXXVII

By means of Taylor's Theorem, show that

1. $\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \cdots$.

2. $\log(a+x) = \log a + \frac{x}{a} - \frac{x^2}{a^2 \cdot 2!} + \frac{2x^3}{a^3 \cdot 3!} + \cdots$.

3. $\cos(a+x) = \cos a - x\sin a - \frac{x^2}{2!}\cos a + \frac{x^3}{3!}\sin a + \cdots$.

4. $e^{x+h} = e^x + he^x + \frac{a^2}{2!}e^x + \frac{a^3}{3!}e^x + \cdots$.

5. $\log\sin(a+x) = \log\sin x + a\cot x - \frac{a^2}{2}\operatorname{cosec}^2 x + \frac{a^3}{3}\cot x \operatorname{cosec}^2 x + \cdots$.

ART. **15. Integration by series.** Determinations of integrals based upon developments into series occur quite often in the applications of the Calculus; we must always have recourse to them when other methods are not available. This mode of finding integrals is known as **integration by series.**

The mode of procedure is as follows: We assume that we are able to develop the function $f(x)$, occurring in the integral

$$\int f(x)\,dx,$$

into a convergent series arranged according to ascending powers of x; thus, the integration of

(1) $$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

gives $$\int f(x)\,dx = \int (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\,dx$$
$$= \int a_0\,dx + \int a_1 x\,dx + \int a_2 x^2\,dx + \cdots,$$

whence

(2) $$\int f(x)\,dx = a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + \cdots + C.$$

We note that the resulting series (2) is undoubtedly convergent for all values of x that make series (1) convergent. We need merely to write it in the form

$$x\left(a_0 + \frac{a_1}{2}x + \frac{a_2}{3}x^2 + \frac{a_3}{4}x^3 + \cdots\right)$$

to see that according to p. 322 the expression in parenthesis is convergent, and hence series (2) also.

We pass at once to some examples. Let the first be

$$\int \frac{dx}{1+x^2}. \tag{3}$$

According to p. 314, when $x^2 < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots, \tag{4}$$

and

$$\int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + x^8 \cdots)\, dx \tag{5}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots + C.$$

We proceed to draw an important conclusion from this equation. According to p. 176, the integral of the left member is equal to arc tan x; we thus obtain the equation

$$\text{arc tan}\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots + C.$$

To determine the constant C, we put $x = 0$; it is sufficient to consider arc tan x to be the *smallest* positive angle whose tangent is x, for when we know one of the angles having a given tangent, we have trigonometric formulæ which enable us readily to determine all others. With this restric-

tion, we have $\operatorname{arc}\tan 0 = 0$, and hence $C = 0$. We have, therefore,

(6) $$\operatorname{arc}\tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots.$$

In this way we have developed $\operatorname{arc}\tan x$ into a series, called **Gregory's Series,*** which, if obtained by Maclaurin's Theorem, would have required complicated computations, since the higher derivatives of $\operatorname{arc}\tan x$ would then have been needed.

If we make $x = 1$, then $\operatorname{arc}\tan 1$ is the length of the arc whose tangent is equal to unity; *i.e.* the arc $\frac{\pi}{4}$. We obtain, accordingly, from equation (4) for $x = 1$,

(5) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,$$

and this series having $\frac{\pi}{4}$ as its limit may be used to determine values differing little at will from the true value of $\frac{\pi}{4}$.

To compute π, it is best to write the series in the form

$$\frac{\pi}{4} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots$$
$$= \frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \frac{2}{13 \cdot 15} + \cdots,$$

and

(6) $$\frac{\pi}{8} = \frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \frac{1}{13 \cdot 15} + \cdots.$$

As a second example, we take the integral

(7) $$\int \frac{dx}{\sqrt{1 - x^2}}.$$

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* James Gregory (1638–1675), an English mathematician, made important contributions to the development of this theory of infinite series. The term *convergent* was introduced by him.

By the Binomial Theorem,

(8)
$$\frac{1}{\sqrt{1-x^2}}=\left\{1+\frac{1}{2}\cdot\frac{x^2}{1}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{x^4}{1\cdot 2}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{x^6}{1\cdot 2\cdot 3}+\cdots\right\},$$

and, after integration,

$$\int\frac{dx}{\sqrt{1-x^2}}=\int dx+\int\frac{1}{2}\cdot\frac{x^2}{1}dx+\int\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{x^4}{1\cdot 2}dx+\cdots+C$$
$$=x+\frac{1}{2}\frac{x^3}{3}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}\frac{x^5}{5}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3}\frac{x^7}{7}+\cdots+C.$$

According to p. 176 the integral of the left member is equal to arc sin x; therefore we obtain the equation

$$\text{arc sin } x=x+\frac{\frac{1}{2}}{1}\frac{x^3}{3}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}\frac{x^5}{5}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3}\frac{x^7}{7}+\cdots+C.$$

To determine the constant we put $x=0$; and find (under the same restriction as was imposed on arc tan x), that $C=0$ also, and hence have

(9) $$\text{arc sin } x=x+\frac{\frac{1}{2}}{1}\frac{x^3}{3}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2}\cdot\frac{x^5}{5}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3}\cdot\frac{x^7}{7}+\cdots.$$

If we put $x=\frac{1}{2}$, then arc sin $x=\frac{\pi}{6}$. We therefore have

(10) $$\frac{\pi}{6}=\frac{1}{2}+\frac{\frac{1}{2}\cdot(\frac{1}{2})^3}{1\cdot 3}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot(\frac{1}{2})^5}{1\cdot 2\cdot 5}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(\frac{1}{2})^7}{1\cdot 2\cdot 3\cdot 7}+\cdots.$$

This series is better adapted to the computation of π than the previous one.

The computation is made still more readily by means of Gregory's Series applied to one of certain trigonometric

relations, of which we give the following, due to Gauss,* as a specimen:

$$(11)\quad \frac{\pi}{4} = 12 \text{ arc tan } \tfrac{1}{18} + 8 \text{ arc tan } \tfrac{1}{57} - 5 \text{ arc tan } \tfrac{1}{239}.$$

The correctness of this relation may be verified by the methods of trigonometry, and it will be necessary to use but a few terms of the series for arc tan x to obtain quite a close approximation to the value of π.

We also see how much superior such methods as these for the calculation of π are to those applied in elementary geometry.

Exercise. By means of (11) calculate the value of π to 8 or more decimal places. Ans. $\pi = 3.141, 592, 653, 589, 793, 238, 462, 643, 383, 279, 502, 884, \cdots$.

ART. **16. Table of series.** We collect into a table the principal series which we have considered:

1. $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \cdots.$
(Maclaurin's Series.)

2. $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \dfrac{h^3}{3!}f'''(x) + \cdots.$
(Taylor's Series.)

3. $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots.$
(The Binomial Series.)

4. $\dfrac{1}{\sqrt{1-x^2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{1\cdot 3}{2\cdot 4}x^4 + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots.$

---

* Carl Friedrich Gauss (1777–1855), the greatest of mathematicians, *princeps mathematicorum*, Professor in the University of Göttingen, enriched all branches of mathematics, both pure and applied, with the lasting fruits of his wonderful genius. His classic work on the Theory of Numbers, *Disquisitiones Arithmeticae*, was published (1801) when he was only twenty-four years of age. With Weber, he invented the first electric telegraph.

5. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$

6. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$

7. $\arcsin x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \cdots.$

8. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

(Gregory's Series.)

9. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$

(The Exponential Series.)

10. $a^x = 1 + x \log a + \frac{x^2}{2!} \log^2 a + \frac{x^3}{3!} \log^3 a + \cdots.$

11. $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

(The Logarithmic Series.)

ART. **17. Indeterminate forms.** We have shown (Footnote, pp. 121–123) that the quotient

$$\frac{\sin x}{x}$$

approaches unity as x approaches zero; this conclusion can be deduced directly by means of our developments into series. We have (p. 329)

$$\frac{\sin x}{x} = \frac{\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}, \tag{1}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots; \tag{2}$$

and when x approaches zero the right member approaches the value 1; *i.e.*

$$\lim_{x \doteq 0} \left[\frac{\sin x}{x}\right] = 1. \tag{3}$$

We may not directly put $x=0$ in the right member of (2), since to obtain (2) we divide by x.

This method may be used to find the limit of other fractions for certain values of the variable, for which the numerator and denominator both are equal to zero. Such a fraction is

$$\frac{1-(1+x)^n}{\log(1+x)};$$

when $x=0$, both numerator and denominator vanish. By the developments into series made on pp. 341 and 338, we find

$$(4)\qquad \frac{1-(1+x)^n}{\log(1+x)}=\frac{1-\left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots\right\}}{\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}+\cdots},$$

$$=\frac{-\frac{n}{1}-\frac{n(n-1)}{1\cdot 2}x-\cdots}{1-\frac{x}{2}+\frac{x^2}{3}+\cdots}.$$

If we now let x approach zero, we see that the fraction on the right approaches $-n$; that is,

$$(5)\qquad \lim_{x\doteq 0}\left[\frac{1-(1+x)^n}{\log(1+x)}\right]=-n.$$

We pass now to the consideration of the fraction

$$(6)\qquad \frac{f(x)}{\phi(x)},$$

and deduce a general method applicable to all such cases.

For convenience, we call the value of x, which causes a fraction to assume the indeterminate form $\frac{0}{0}$, a **critical value** of x.

For all values of x, except the critical values, the fraction has a definite value which can be determined by direct substitution of the value of x; for the critical values of x, this is, however, not the case, and, as in the instances above, so, in general, we seek to determine the *limit* which the fraction approaches as x approaches the critical value.

We assume that both functions vanish when $x = a$, so that

$$f(a) = 0, \quad \phi(a) = 0; \tag{7}$$

and, accordingly, $x = a$ is a critical value.

By Taylor's Theorem,

$$\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{1 \cdot 2} f''(a) + \cdots, \\ \phi(a+h) &= \phi(a) + h\phi'(a) + \frac{h^2}{1 \cdot 2} \phi''(a) + \cdots; \end{aligned} \tag{8}$$

and, inasmuch as $f(a) = 0$ and $\phi(a) = 0$,

$$\frac{f(a+h)}{\phi(a+h)} = \frac{hf'(a) + \dfrac{h^2}{1 \cdot 2} f''(a) + \cdots}{h\phi'(a) + \dfrac{h^2}{1 \cdot 2} \phi''(a) + \cdots},$$

$$= \frac{f'(a) + \dfrac{h}{1 \cdot 2} f''(a) + \cdots}{\phi'(a) + \dfrac{h}{1 \cdot 2} \phi''(a) + \cdots}. \tag{9}$$

This expression is true for all values of h (except $h = 0$) for which Taylor's Series converges. If we let $h \doteq 0$, then

$$\lim_{x \doteq a} \left[\frac{f(x)}{\phi(x)} \right] = \frac{f'(a)}{\phi'(a)}. \tag{10}$$

We see thus that *we can find the required limit of the fraction by simply substituting for the numerator and denominator their derivatives with respect to* x.

ART. **18. Illustrative examples of the determination of the limits of indeterminate forms.**

I. To find the limit of

$$(11) \qquad \frac{x^3 - 6x^2 + 11x - 6}{x^3 + 2x^2 - x - 2}$$

when $x = 1$.

By substituting 1 for x we see that the fraction assumes the form $\frac{0}{0}$. We have

$$f'(x) = 3x^2 - 12x + 11,$$

$$\phi'(x) = 3x^2 + 4x - 1,$$

whence $\qquad f'(1) = 2$ and $\phi'(1) = 6$,

so that $\qquad \dfrac{f'(1)}{\phi'(1)} = \dfrac{2}{6} = \dfrac{1}{3};$

i.e. (12) $\qquad \lim\limits_{x \doteq 1} \left[\dfrac{x^3 - 6x^2 + 11x - 6}{x^3 + 2x^2 - x - 2}\right] = \frac{1}{3}.$

II. To find the limiting value of

$$(13) \qquad \frac{f(x)}{\phi(x)} = \frac{1 - x}{\log x},$$

when $x = 1$. We have

$$f'(x) = -1, \quad \phi'(x) = \frac{1}{x};$$

consequently,

$$(14) \qquad \lim_{x \doteq 1} \left[\frac{1 - x}{\log x}\right] = \frac{-1}{1} = -1.$$

III. For the fraction $\dfrac{f(x)}{\phi(x)} = \dfrac{x^2 - 1}{\log x}$,

we find $\qquad f'(x) = 2x, \quad \phi'(x) = \dfrac{1}{x},$

and $\qquad \lim\limits_{x \doteq 1} \left[\dfrac{x^2 - 1}{\log x}\right] = \dfrac{2x}{x} = 2.$

The development of functions into series is of great help also in the determination of the limits of fractions in which the numerator and the denominator are infinite, as well as of products, one of whose factors is zero, and the other is infinite for the critical value.

IV. To determine (p. 241)

$$(15)\qquad \lim_{a \doteq b}\left[\frac{1}{a-b}\log\frac{(a-x)b}{(b-x)a}\right].$$

When $a=b$, $\dfrac{1}{a-b}$ becomes infinite, and

$$(16)\qquad \log\frac{(a-x)b}{(b-x)a}=\log 1=0.$$

We put

$$a-x=a\left(1-\frac{x}{a}\right)\text{ and } b-x=b\left(1-\frac{x}{b}\right),$$

so that

$$\log\frac{(a-x)b}{(b-x)a}=\log\frac{ab\left(1-\frac{x}{a}\right)}{ab\left(1-\frac{x}{b}\right)}=\log\frac{1-\frac{x}{a}}{1-\frac{x}{b}}$$

$$=\log\left(1-\frac{x}{a}\right)-\log\left(1-\frac{x}{b}\right),^{*}\text{ or (p. 338)}$$

$$\log\frac{(a-x)b}{(b-x)a}=-\left\{\frac{x}{a}+\frac{1}{2}\frac{x^2}{a^2}+\frac{1}{3}\frac{x^3}{a^3}+\cdots\right\}+\left\{\frac{x}{b}+\frac{1}{2}\frac{x^2}{b^2}+\frac{1}{3}\frac{x^3}{b^3}+\cdots\right\}$$

$$=x\left(\frac{1}{b}-\frac{1}{a}\right)+\frac{x^2}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\frac{x^3}{3}\left(\frac{1}{b^3}-\frac{1}{a^3}\right)+\cdots$$

$$=x\frac{a-b}{ab}+\frac{x^2}{2}\,\frac{a^2-b^2}{a^2b^2}+\frac{x^3}{3}\,\frac{a^3-b^3}{a^3b^3}+\cdots$$

$$=(a-b)\left\{\frac{x}{1}\cdot\frac{1}{ab}+\frac{x^2}{2}\,\frac{a+b}{a^2b^2}+\frac{x^3}{3}\,\frac{a^2+ab+b^2}{a^3b^3}+\cdots\right\}.$$

---

* Formula 6, Appendix.

Substituting this value in (15), we have

$$\frac{1}{a-b}\log\frac{(a-x)b}{(b-x)a}=\frac{x}{1}\cdot\frac{1}{ab}+\frac{x^2}{2}\frac{a+b}{a^2b^2}+\frac{x^3}{3}\frac{a^2+ab+b^2}{a^3b^3}+\cdots,$$

and hence

$$\lim_{a \doteq b}\left[\frac{1}{a-b}\log\frac{(a-x)}{(b-x)}\frac{b}{a}\right]=\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\cdots$$

$$=\frac{x}{a^2}\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\cdots\right),$$

and by applying the formula for the sum of a geometric series (p. 314),

$$(17)\qquad \lim_{a \doteq b}\left[\frac{1}{a-b}\log\frac{(a-x)\,b}{(b-x)\,a}\right]=\frac{x}{a^2}\frac{1}{1-\frac{x}{a}}=\frac{x}{a(a-x)}.$$

V. If, in the expression

$$u=\frac{1}{x}-\frac{1}{\log(1+x)},$$

x approach the value zero, then both minuend and subtrahend become infinite, and the expression is indeterminate. In order to find its limiting value, we first reduce to a common denominator,

$$u=\frac{\log(1+x)-x}{x\log(1+x)}.$$

This assumes the form $\frac{0}{0}$ when $x=0$. The limit of this fraction could be found by the method which we have established for the form $\frac{0}{0}$. Leaving this to the student as an exercise, we determine the limit by using the expansion of

$\log(1+x)$ into a series. When this is substituted, we obtain

$$u = \frac{\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \cdots - x}{x\left(\frac{x}{1} - \frac{x^2}{2} \cdots\right)} = \frac{-\frac{x^2}{2} + \frac{x^3}{3} \cdots}{x\left(\frac{x}{1} - \frac{x^2}{2} \cdots\right)};$$

and dividing numerator and denominator by x^2, and then letting $x \doteq 0$, we find the required limiting value

$$(18) \qquad \lim_{x \doteq 0}\left[\frac{1}{x} - \frac{1}{\log(1+x)}\right] = -\frac{1}{2}.$$

VI. To find

$$\lim_{x \doteq \infty}\left[\frac{e^x}{x}\right].$$

When x grows large without bound, the numerator does so likewise. But by the series for e^x, we have

$$\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{x}{2\,!} + \frac{x^2}{3\,!} + \frac{x^3}{4\,!} + \cdots,$$

and this series is infinite when x is infinite; that is,

$$(19) \qquad \lim_{x \doteq \infty}\left[\frac{e^x}{x}\right] = \infty.$$

If we now put $e^x = y$, *i.e.* $x = \log y$, then $y \doteq \infty$, when $x \doteq \infty$, and

$$(20) \qquad \lim_{y \doteq \infty}\left[\frac{y}{\log y}\right] = \infty,$$

and hence

$$(21) \qquad \lim_{y \doteq \infty}\left[\frac{\log y}{y}\right] = 0.$$

When, then, y grows larger and larger, the quotient of $\log y$ by y approaches more and more nearly to zero.

If, in the last equation, we put

$$y = \frac{1}{x},$$

as $y \doteq \infty$, $x \doteq 0$, and we have

$$\frac{\log y}{y} = x \log \frac{1}{x} = -x \log x,$$

and
$$\lim_{x \doteq 0} [-x \log x] = \lim_{y \doteq \infty} \left[\frac{\log y}{y}\right] = 0;$$

here we have a formula for the limiting value of the product $x \log x$, one of whose factors, x, approaches zero, while the other, $\log x$, grows large without bound.

VII. We apply this result in determining

$$\lim_{x \doteq 0} [x^x],$$

which for the critical value assumes the form 0^0.

We have *

$$x^x = e^{x \log x}.$$

Accordingly,

$$\lim_{x \doteq 0} [x^x] = \lim_{x \doteq 0} [e^{x \log x}] = e^{\lim_{x \doteq 0} [x \log x]} = e^0 = 1.$$

VIII. Consider, next,

$$\lim_{x \doteq 0} \left[\frac{1}{x}\right]^{\sin x}.$$

This is of the form ∞^0 if $x = 0$. We transform it by noticing that $\frac{1}{x} = e^{-\log x}$, hence we seek to find $\lim_{x \doteq 0} e^{-\log x \sin x}$,

or
$$e^{-\lim_{x \doteq 0} [\log x \sin x]}.$$

---

* Formula 7, Appendix.

But $\lim_{x \doteq 0} \log x \sin x = \lim_{x \doteq 0} x \frac{\sin x}{x} \log x$

$$= \lim_{x \doteq 0} x \log x, \left(\text{since} \lim_{x \doteq 0} \frac{\sin x}{x} = 1\right)$$

$$= 0, \text{ as shown above.}$$

Accordingly,

$$\lim_{x \doteq 0} \left[\frac{1}{x}\right]^{\sin x} = e^0 = 1.$$

IX. Let us take up next

$$\lim_{x \doteq 1} x^{\frac{1}{1-x}}.$$

If $x = 1$, this is of the form 1^∞.

Let $$y = x^{\frac{1}{1-x}}.$$

Then $\log y = \frac{\log x}{1 - x}$. For $x = 1$, this is of the form $\frac{0}{0}$. Accordingly,

$$\lim_{x \doteq 1} [\log y] = \lim_{x \doteq 1} \left[\frac{\log x}{1 - x}\right] = -1.$$

$$\therefore \lim_{x \doteq 1} y = \frac{1}{e}.$$

ART. **19. Types of indeterminate forms.** The principal indeterminate forms are

$$\frac{0}{0}; \quad \frac{\infty}{\infty}; \quad 0 \cdot \infty; \quad \infty - \infty; \quad 0^0; \quad \infty^0; \quad 1^\infty;$$

all of these have been exemplified above.

The form occurring most often and treated most simply is the first. Generally the second and third forms may be

reduced to the first form by a simple transformation. Any fraction $\frac{A}{B}$ may be written in the form $\frac{\frac{1}{B}}{\frac{1}{A}}$. If A and B increase without bound, $\frac{1}{A}$ and $\frac{1}{B}$ approach zero, so that the indeterminate form becomes $\frac{0}{0}$. Also, if in a product AB, one of the factors, A, approaches zero, while the other, B, increases without bound, we may write the product in the form $\frac{A}{\frac{1}{B}}$, which is in the first form, *viz.*, $\frac{0}{0}$.

Usually functions in the form $\frac{0}{0}$ can be evaluated by differentiating the numerator and the denominator, and substituting the critical value in the quotient of the results. Sometimes, however, this will not succeed, as the function retains the form $\frac{0}{0}$, no matter how often the differentiation is repeated. The function $\frac{x^{-1}}{e^{-x}}$, as $x \doteq \infty$, is an instance. In such cases the evaluation may often be accomplished by expansion into series.

It may be proved without much difficulty that the form $\frac{\infty}{\infty}$ can, like $\frac{0}{0}$, be evaluated by substituting the critical value in the quotient of the derivative of the numerator by that of the denominator, but we let this simple mention of this theorem suffice.

EXERCISES XXXVIII

Find the limits of the following expressions:

1. $\frac{x^2 - 25}{x - 5}$, as $x \doteq 5$.
2. $\frac{x^3 - a^3}{x - a}$, as $x \doteq a$.
3. $\frac{x^4 - r^4}{x^3 - r^3}$, as $x \doteq r$.
4. $\frac{6x^2 + x - 35}{10x^2 + 29x + 10}$, as $x \doteq -\frac{5}{2}$.

5. $\frac{\tan x}{x}$, as $x \doteq 0$.

6. $\frac{1-x^n}{1-x}$, as $x \doteq 1$.

7. $\frac{x^5 - 5x^2 + 4}{x^6 + 2x^5 + 3x - 6}$, as $x \doteq 1$.

8. $\frac{x}{a^x - 1}$, as $x \doteq 0$.

9. $\frac{\tan x}{\sin x}$, as $x \doteq 0$.

10. $\frac{1-\cos x}{x}$, as $x \doteq 0$.

11. $\frac{1-\cos x}{x^2}$, as $x \doteq 0$.

12. $\frac{1-\cos x}{\sqrt{x}}$, as $x \doteq 0$.

13. $x \cot x$, as $x \doteq 0$.

14. $\frac{x^{-1}}{\cot x}$, as $x \doteq 0$.

15. $\frac{x^3 + x^2 - 4x - 4}{x^4 + 2x^3 - 3x^2 - 8x - 4}$.

 i. as $x \doteq -2$.

 ii. as $x \doteq +2$.

 iii. as $x = -1$.

16. $\frac{\log x}{x-1}$, as $x \doteq 1$.

17. $\frac{e^x}{x^3}$, as $x \doteq \infty$.

18. $\frac{a^x - b^x}{x}$, as $x \doteq 0$.

19. $(\cos x)^{\frac{1}{x}}$, as $x \doteq 0$.

20. $xe^{\frac{1}{x}}$, as $x \doteq 0$.

21. $\frac{e^x - e^{\sin x}}{x - \sin x}$ as $x \doteq 0$.

22. $\sin(x+1) \cdot \frac{5x+1}{x+1}$, as $x \doteq -1$.

ART. **20. Calculation with small quantities.** An important practical application of the expansion of functions into series, given in this chapter, occurs in *calculation with small quantities;* in such cases it is generally sufficient to take only the first few terms of the series, so that a simple and easily handled expression is obtained from the originally infinite series.

But we must have a clear conception of what is meant by small quantities. "Absolutely small" quantities are non-existent as well for the investigator in physical science as for the mathematician. If we endeavor to determine the true capacity of a liter flask by weighing it when full of water and when empty, a determination to a ten thousandth, *i.e.* weighing accurately within $\frac{1}{10}$ gram, is in most cases

sufficient, and we may regard the last weight as a small quantity. In careful chemical analyses, however, where tenths of a milligram are of the greatest importance, an error of $\frac{1}{10}$ gram would make the analysis worthless. The astronomer in measuring the distances of planets can neglect lengths of many kilometers as in nowise affecting the accuracy of his results, while the physicist in measuring the lengths of light waves finds millionths of millimeter of decisive importance in observations and calculation. "Small Quantity" is, therefore, a relative conception, and *we have the right to call a quantity small only in comparison with a second much larger one.* We may never neglect a quantity in our calculations because it appears to be small in itself (being only a millionth, say), but can do so only when it occurs in connection with a quantity so much larger that the small quantity would exert no influence upon the degree of accuracy which we wish to attain. In order to express any given quantity as the sum of one or two large quantities, and of such negligible small quantities, the developments into series often give us valuable assistance, as we shall show in some examples.

ART. **21. Reduction of barometric readings to 0° C.** The length l of a column of mercury sustained by the atmosphere, the cross-section being constant, varies with the temperature according to the formula

$$l = l_0(1 + 0.00018\,t),$$

where l_0 denotes the length at the temperature $t = 0°$. The barometric height l_0 corresponding to the height l observed at the temperature t would, therefore, be determined by the formula,

$$l_0 = \frac{l}{1 + 0.00018\,t}.$$

But according to p. 314,

$$\frac{1}{1+\alpha}=1-\alpha+\alpha^2-\cdots,$$

or, if $\alpha=0.00018\,t$,

$$l_0=l(1-0.00018\,t+[0.00018\,t]^2\cdots).$$

Now even if $t=30°$, the third term of the series $[0.00018\,t]^2$ is smaller than 0.00003, which may be neglected in comparison with unity, so that we may write as a quite sufficient approximation,

$$l_0=l(1-0.00018\,t).$$

It is generally advantageous to transform the equations so as to make the small quantities appear as terms of a sum in connection with unity. If a calculation or observation is to be carried out accurately within one-tenth of one per cent, terms whose aggregate is less than 0.001 may be neglected; if an error of two or three per cent is admissible, terms less than 0.01 may be neglected, etc., etc.

ART. **22.** **Simplified hypsometric formula.** We found (p. 223) that the elevation H above the earth's surface was

$$(1)\qquad H=\frac{B}{S}\log\frac{B}{y};$$

even at elevations of 1000 meters, B is but slightly greater than y, so that $B-y$ may be regarded as small in comparison with B as well as with y. If we put equation (1) in the form

$$H=\frac{B}{S}\log\left(1+\frac{B-y}{y}\right),$$

and expand according to p. 338, we have

$$(2)\qquad H=\frac{B}{S}\cdot\frac{B-y}{y}\left(1-\frac{B-y}{2y}\right);$$

the terms involving higher powers of the small quantity $\frac{B-y}{y}$ being neglected; in many cases we may even neglect the second term within the parenthesis.

We can also write equation (1) in the form

$$H = -\frac{B}{S}\log\left(1 - \frac{B-y}{B}\right),$$

and obtain on developing into series,

$$(3) \qquad H = \frac{B}{S}\cdot\frac{B-y}{B}\left(1 + \frac{(B-y)}{2B}\right).$$

This formula also can be employed for moderate elevations; but we can get a much better approximation by the aid of the following artifice. In formula (2) the corrective term is negative, and in formula (3) it is positive, while in either case it is about the same $\left(\frac{B-y}{2y}\right.$ is but little different from $\left.\frac{B-y}{2B}\right)$; the true value lies therefore about midway between

$$\frac{B}{S}\cdot\frac{B-y}{y} \text{ and } \frac{B}{S}\cdot\frac{B-y}{B}.$$

The two expressions differ only in the denominator, and if we introduce the average denominator $\frac{y+B}{2}$, we have

$$(4) \qquad H = 2\frac{B}{S}\cdot\frac{B-y}{B+y},$$

which is the formula most used in practice.

CHAPTER XI

MAXIMA AND MINIMA

ART. 2. **Conditions for a maximum or minimum.** The accompanying curve (Fig. 60), which corresponds to the equation

(1) $$y = \sin x,$$

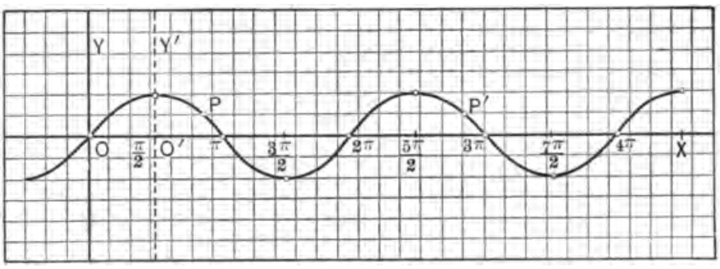

FIG. 60.

reaches its highest position in the points whose abscissæ have the values

$$\frac{\pi}{2},\ \frac{5\pi}{2},\ \frac{9\pi}{2},\ \cdots,$$

and its lowest position where the values of x are

$$-\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{7\pi}{2},\ \cdots;$$

the first-named positions are called **maxima,** and the second, **minima,** of the curve. The function $\sin x$ has therefore maximum values when x is equal to

$$\frac{\pi}{2},\ \frac{\pi}{2} \pm 2\pi,\ \frac{\pi}{2} \pm 4\pi,\ \cdots,$$

and minimum values when x is equal to

$$\frac{3\pi}{2},\quad \frac{3\pi}{2}\pm 2\pi,\quad \frac{3\pi}{2}\pm 4\pi,\quad \cdots.$$

At all of the points having these abscissæ the tangent to the curve is parallel to the x-axis; therefore, for all these values of x the coefficient m (p. 25) of the tangent line is zero; that is, $\cos x$ must be zero for these values of x, which in fact is the case.

These considerations may be extended to any curve corresponding to an equation of the form

$$(2) \qquad y = f(x).$$

We define maxima and minima formally as follows:

If h denote a fixed number as small as may be necessary, and if as x increases from $a - h$ to a, $y = f(x)$ also increases, and as x increases from a to $a + h$, y decreases, then $x = a$ is the abscissa of a ***maximum*** *point of y. Likewise, if as x increases from $a - h$ to a, y decreases, and as x increases from a to $a + h$, y increases, then $x = a$ is the abscissa of a* ***minimum*** *point of y.*

At every position where a curve has a maximum or minimum the tangent to the curve is parallel* to the axis of

* This is true only when the function and its first derivative are continuous. If the first derivative becomes infinite, the tangent is perpendicular to the x-axis, and there may be a maximum or minimum. In case a maximum or minimum exists, the point has other and more characteristic properties than those of maxima and minima, and it is accordingly usually not classified with maxima and minima. There may also be a maximum or minimum if the first derivative is discontinuous without becoming infinite. By turning Fig. 45, p. 164, about O as a pivot, through a negative angle whose magnitude is greater than α and less than α', the point P becomes a minimum. Since we have restricted ourselves to the consideration of continuous functions only, the closer examination of these points does not fall within the scope of our work.

abscissæ, and hence the slope of the tangent is equal to zero; but it must be noted that the converse is not always true. Accordingly, for all these values of x,

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x) = 0. \tag{3}$$

This is the equation from which the values of x are calculated, for which $f(x)$ *may* have a maximum or minimum.

As an example, the derivative of the function

$$y = 2x^3 - 9x^2 + 12x - 1,$$

which is represented by the accompanying curve (Fig. 61), is

$$\frac{dy}{dx} = 6x^2 - 18x + 12.$$

Equating this to zero, we have

$$6(x^2 - 3x + 2) = 0.$$

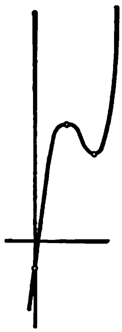

FIG. 61.

The roots of this quadratic equation are $x_1 = 1$ and $x_2 = 2$; the curve shows that for the first value y presents a maximum, for the second a minimum.

Inasmuch as usually only the functions themselves, and not the curves representing them, are known, it is further necessary to ascertain whether the various values of x which satisfy equation (3) actually correspond to a maximum or minimum of the function or not. Inspection of Fig. 62 shows that the curve is concave toward the x-axis at the maximum and convex at the mini-

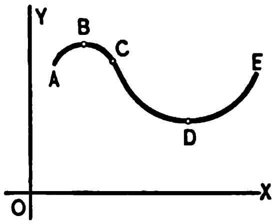

FIG. 62.

mum. Hence, as has been shown previously (p. 278) for any value of x at which a maximum occurs,

$$\frac{d^2y}{dx^2} < 0 \quad \text{or} \quad \frac{d^2}{dx^2}f(x) < 0,$$

and for such at which a minimum occurs,

$$\frac{d^2y}{dx^2} > 0 \quad \text{or} \quad \frac{d^2}{dx^2}f(x) > 0.$$

In the figure, the maximum and the minimum were both above the x-axis, but it may be seen similarly that these conditions hold also for maxima and minima which lie below the x-axis.

The abscissa, x, of a maximum or a minimum must therefore satisfy one of the following sets of conditions:

At a minimum, $\frac{d}{dx}f(x) = 0$; $\frac{d^2}{dx^2}f(x)$, **positive.**

At a maximum, $\frac{d}{dx}f(x) = 0$; $\frac{d^2}{dx^2}f(x)$, **negative.**

It may happen that both the first and the second derivative assume the value zero for the same value of x. This case will be discussed later (p. 367).

In the example above we find

$$y'' = \frac{d^2y}{dx^2} = 6(2x - 3)$$

and for $x = 1$, $y'' = -6$, while for $x = 2$, $y'' = 6$. Hence, $x = 1$ corresponds to a maximum, and $x = 2$ to a minimum, just as the figure indicates.

ART. 2. **Points of inflexion of curves.** We have already defined (p. 277) a **point of inflexion** as a point separating a concave from a convex portion of a curve. At such a

point the curve is crossed by the tangent so that it lies partly on the one side, partly on the other of the tangent. The tangent at a point of inflexion is often called an **inflexional tangent.**

To fix our ideas, we assume that the curve is concave toward the x-axis before the point of inflexion, and convex toward the x-axis afterward; as, for instance, the curve in Fig. 63; *i.e.* from A to C the slope of the tangent decreases continually (the angle τ itself first diminishes along AB through positive acute angles to zero, then along BC it diminishes from 180° to the angle CFX, the slope of the inflexional tangent); along the convex portion from C to E, $\tan \tau$ continually increases, the angle τ increasing from CFX to 180°, and then from zero through acute angles the tangent accordingly assumes a minimum value at C. Just the opposite would be true if a concave portion of the curve were joined on at E; E would likewise be a point of inflexion, but at this point the slope of the tangent line would assume a maximum value. It is easily seen that either one or the other of these cases must always occur when a curve has a point of inflexion.

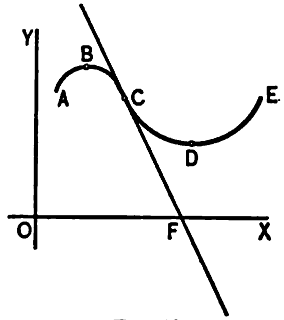

FIG. 63.

To find the points which may be points of inflexion of a curve, we have to determine such values of x as will make the slope of the tangent (which is equal to the first derivative) a maximum or minimum; that is, the values for which

$$\frac{d \tan y}{dx} = \frac{dy'}{dx} = \frac{d^2y}{dx^2} = 0.$$

In words, *those values of x which cause the second derivative to vanish (and no others) may be abscissæ of points of inflexion.*

To determine whether there actually is a point of inflexion for these values of x, we must examine the third derivative (*i.e.* the *second* derivative of the slope for which we are seeking the maxima and minima). If the third derivative does not become zero for the value of x in hand, this value corresponds to a point of inflexion.

For the equation of the curve considered above,

$$y = 2x^3 - 9x^2 + 12x + 1,$$

we find

$$\frac{dy}{dx} = 6(x^2 - 3x + 2); \quad \frac{d^2y}{dx^2} = 6(2x - 3).$$

The abscissa of the only point which can be a point of inflexion is found by solving the equation

$$2x - 3 = 0,$$

which gives $x = \frac{3}{2}$, to which $y = \frac{11}{2}$ corresponds as ordinate. These are the values of the coördinates of the point of inflexion (Fig. 61).

It may also be noticed that if x increases continuously, $\frac{dy}{dx}$ changes sign from positive to negative as y passes through a maximum value, and from negative to positive as y passes through a minimum value; but if $\frac{dy}{dx}$ does not change sign, y has neither a maximum nor minimum.

We have already mentioned (Footnote, p. 362) that sometimes y has a maximum or minimum value when $\frac{dy}{dx}$ changes sign by passing through infinity; such cases, however, demand special investigation, which the scope of our work does not permit us to make.

EXERCISES XXXIX

Examine the following curves for points of inflexion:

1. $y = \dfrac{x^3}{a^2 + x^2}$. *Ans.* $x = 0, \pm a\sqrt{3}$.

(The abscissa only of the point is given in each answer.)

2. $y = \dfrac{x^4}{a^3}$. *Ans.* $x = 0$.

3. $y = 3x^2 - x^3$. *Ans.* $x = 1$.

4. $y = \sin x$. *Ans.* $x = 0, \pi$, etc.

5. $y^2 = 2px$. *Ans.* None.

6. $y = \tan x$. *Ans.* $x = 0, \pi \cdots$.

7. $y = (\log x)^3$. *Ans.* $x = e^2$.

ART. 3. **Exceptional cases; general theory.** The exceptional case that both the first and the second derivative vanish for the abscissa which we are examining, will be treated best by developing the criteria for maxima and minima from the definition in a more general manner.

The definition may be expressed in symbols as follows:

The differences

$$f(x + h) - f(x) \text{ and } f(x - h) - f(x)$$

have like signs at a maximum or minimum (negative at a maximum and positive at a minimum), while they have unlike signs at a point which is neither a maximum nor a minimum.

By Taylor's Theorem,

$$(1)\quad f(x + h) - f(x) = hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{\text{iv}}(x) + \cdots.$$

$$(2)\quad f(x - h) - f(x) = -hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{\text{iv}}(x) - \cdots.$$

We now apply the following theorem:

For an infinite series of increasing positive integral powers of some variable, as h (convergent for the values of h to be used), there exists a positive number H, such that for all values of h numerically less than H, the first term of the series is numerically greater than the aggregate of the others.

(If we take a very small number as H, the theorem seems plausible, since the higher powers of H would be exceedingly small in comparison with the lowest power of H occurring. For a *proof* of the theorem we refer to works on algebra.)

By means of this theorem we see that if $f'(x)$ is not zero for the value of x in question, then for sufficiently small values of h, the signs of (1) and (2) will be determined by the first terms respectively, and will hence be unlike.

Introducing, for this discussion only, the abbreviation Mm. for "Maximum or Minimum," we have just seen that there can be no Mm. for any value of x, for which $f'(x) \neq 0$, and we have as a necessary, though not sufficient, condition for a Mm. at x, $f'(x) = 0$.

If $f'(x) = 0$, the two series will begin with the terms in h^2. If $f''(x) \neq 0$, the signs of the series will be like, and there is a Mm.

If, however, $f''(x) = 0$ also, the series will begin with the terms in h^3. If $f'''(x) \neq 0$, the signs will be unlike when h is sufficiently small, and there is no Mm. If $f'''(x) = 0$ likewise, but $f^{\text{iv}}(x) \neq 0$, there is a Mm. Proceeding in this way, we reach the following general result:

If the rth derivative, $f^{(r)}(x)$, is the first of the successive derivatives which does not become zero for the value of x in question, then if r is odd, x does not correspond to a Mm.; but if r is even, there is a Mm. for this value of x, a maximum if $f^{(r)}(x)$ is negative, and a minimum if it is positive.

It follows readily that when r is odd, and greater than unity, x is the abscissa of a point of inflexion.

ART. 4. Collected criteria concerning forms of curves. We collect into a table the principal results which we have established relative to the forms of curves for a given abscissa. We consider the curve whose equation is

$$y = f(x),$$

and the table indicates what is the shape of the curve, if for any given abscissa, the various derivatives assume the values specified. With the exception of the first, the conditions indicated are sufficient but not necessary. The marks, ···, mean that the value which the derivative may assume is immaterial. In the table, as throughout our treatment, we exclude the case that any of the derivatives become infinite, for the value of x in hand.

TABLE OF CRITERIA FOR CURVES

	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	
1.	0	...	...	...	**Tangent parallel to x-axis.**
2.	0	$-$	...	...	**Maximum.**
3.	0	$+$	...	...	**Minimum.**
4.	...	0	$\neq 0$	...	**Point of inflexion.**
5.	0	0	0	$-$	**Maximum.**
6.	0	0	0	$+$	**Minimum.**
7.	$\neq 0$	$+$	...	...	**Convex downward.**
8.	$\neq 0$	$-$	...	...	**Concave downward.**

ART. 5. **Examples of maxima and minima.** I. *To divide a given length a into two segments x and a − x, whose product shall be as great as possible.* Let y denote this product, so that

(1) $$y = x(a - x).$$

Forming the derivatives, we have

(2) $$\frac{dy}{dx} = a - 2x, \quad \frac{d^2y}{dx^2} = -2,$$

and putting $\frac{dy}{dx}$ equal to zero, we find

(3) $$0 = a - 2x, \quad \text{or} \quad x = \frac{a}{2}.$$

That is, at the point whose abscissa is $\frac{a}{2}$, $\frac{dy}{dx}$ is zero, while $\frac{d^2y}{dx^2}$ is not zero, but is negative, and we have a maximum.

II. *Two sides of a triangle being given, to find the included angle for which the area of the triangle is a maximum.*

Let the two sides be denoted by a and b, the angle by x, and the area by y. Then,*

(4) $$y = \tfrac{1}{2}\, ab \sin x.$$

The first derivative is

(5) $$\frac{dy}{dx} = \tfrac{1}{2}\, ab \cos x;$$

hence at a maximum,

(6) $\tfrac{1}{2}\, ab \cos x = 0$; whence $\cos x = 0$, or $x = \frac{\pi}{2}$.

The second derivative is

(7) $$\frac{d^2y}{dx^2} = -\tfrac{1}{2}\, ab \sin x;$$

and, since $\sin \frac{\pi}{2} = 1$, the value of the second derivative at the point whose abscissa is $\frac{\pi}{2}$, is

$$x = -ab,$$

and a maximum occurs at that point.

Therefore, the triangle will have the maximum area when the included angle is a right angle.

III. *To find the base and altitude which a rectangle of given area A must have in order that the perimeter P shall be a minimum.*

If the base be denoted by x, then the altitude is $\frac{A}{x}$, and the perimeter is

(8) $$P = 2x + 2 \cdot \frac{A}{x} = 2\left(x + \frac{A}{x}\right).$$

---

* Formula 55, Appendix.

Differentiating, and putting the first derivative equal to zero, we have

$$(9) \qquad 2\left(1-\frac{A}{x^2}\right)=0,$$

or,
$$x^2 = A, \quad x=\sqrt{A}.$$

Forming the second derivative we find that it is negative for $x=\sqrt{A}$, and therefore the square has the least perimeter of all rectangles of the same area.

IV. *To find the shortest straight line that can be drawn through a given point A, within a right angle, to the sides of the right angle* (Fig. 64).

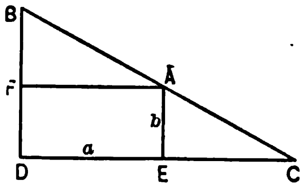

FIG. 64.

Let the required line be denoted by BC, the coördinates of the point $DE=a$ and $AE=b$, and the lines DC and BD by x and y, respectively.

From the two similar triangles BDC and AEC we have the proportion

$$y : x :: b : (x-a),$$

whence
$$y=\frac{bx}{x-a};$$

and
$$\overline{BC}^2 = x^2+\frac{b^2x^2}{(x-a)^2}.$$

The first derivative is

$$(10) \quad \frac{d(\overline{BC}^2)}{dx} = 2\,x+\frac{(x-a)^2\cdot 2\,b^2x - b^2x^2\cdot 2(x-a)}{(x-a)^4};$$

and at a minimum we must have

$$(11) \qquad 2\,x+\frac{(x-a)^2\cdot 2\,b^2x - b^2x^2\cdot 2(x-a)}{(x-a)^4}=0,$$

or
$$2\,x[(x-a)^3 - ab^2]=0.$$

Hence, by solving for x, either

$$x = 0, \text{ or } x = a + \sqrt[3]{ab^2},$$

and the position of the point C for a line BC, which may be a minimum, is thus determined in terms of the coördinates of the given point.

We leave the examination of the second derivative as an exercise for the student. Each value proves to be a minimum.

V. *What sector of a given circle will form the convex surface of a cone of maximum volume?*

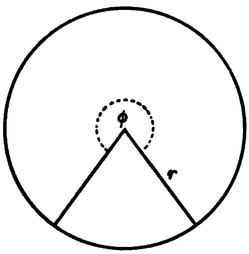

Fig. 65.

Let the radius of the given circle be r (Fig. 65), and let the angle of the sector forming the surface of the cone be ϕ (in circular measure). The arc of the sector is then ϕr, and this is the perimeter of the base of the cone. Denoting the radius of the base of the cone by R, we have

$$2\pi R = \phi r, \text{ or } R = \frac{\phi r}{2\pi}.$$

If h be the altitude of the cone, its volume is

(12) $$V = \frac{\pi R^2 h}{3}.\text{*}$$

Since h, R, and the slant height r of the cone form a right-angled triangle,

$$h = \sqrt{r^2 - R^2} = r\sqrt{1 - \frac{\phi^2}{4\pi^2}}.$$

* Formula 67, Appendix.

Substituting in equation (12), we have

$$V=\frac{\pi}{3}\frac{\phi^2r^3}{4\pi^2}\sqrt{1-\frac{\phi^2}{4\pi^2}}=\frac{r^3\phi^2}{12\pi}\sqrt{1-\frac{\phi^2}{4\pi^2}}, \tag{14}$$

and we now have to find what value of ϕ makes V a maximum. Any value of ϕ which makes V a maximum will also make $\frac{12\pi}{r^3}$ times V a maximum; that is,

$$V_1=\frac{12\pi}{r^3}V=\phi^2\sqrt{1-\frac{\phi^2}{4\pi^2}} \tag{15}$$

will be a maximum whenever V is such. It is therefore sufficient to examine V_1 for maxima or minima.* The equation of condition for ϕ is

$$\frac{dV_1}{d\phi}=2\phi\sqrt{1-\frac{\phi^2}{4\pi^2}}-\phi^2\frac{\frac{\phi}{4\pi^2}}{\sqrt{1-\frac{\phi^2}{4\pi^2}}}=0. \tag{16}$$

Multiplying through by $\sqrt{1-\frac{\phi^2}{4\pi^2}}$ (this is never zero, for ϕ manifestly cannot have the value 2π), we obtain

$$2\phi\left(1-\frac{\phi^2}{4\pi^2}\right)-\frac{\phi^3}{4\pi^2}=0. \tag{17}$$

Dividing by ϕ, we have

$$2-3\frac{\phi^2}{4\pi^2}=0,$$

or

$$\phi=2\pi\sqrt{\tfrac{2}{3}}. \tag{18}$$

* In general, C being any constant, the maxima and minima of $Cf(x)$ are the same as those of $f(x)$. For in the one case we have to solve the equation $Cf'(x)=0$ and in the other $f'(x)=0$, and both these equations have the same roots.

The corresponding angle in degrees is approximately $x = 294°$.

The volume of the maximum cone is

(19) $$V = \frac{2\pi r^3}{9}\sqrt{\tfrac{1}{3}}.$$

That the value of ϕ actually corresponds to a maximum may be seen as follows without examining the second derivative. If $\phi = 2\pi$, the cone is the circle itself and its *volume* is zero; if $\phi = 0$, the cone is a straight line (the radius) and its volume is likewise zero. Between these two zero volumes there must be at least one maximum, and as the first derivative vanishes for only one value of ϕ between zero and 2π, that value determines the maximum.

VI. The following is an example of a function whose second derivative vanishes simultaneously with the first derivative:

(20) $$y = x^3 - 3x^2 + 3x + 2.$$

The first and second derivatives are, respectively,

$$\frac{dy}{dx} = 3x^2 - 6x + 3$$

and $$\frac{d^2y}{dx^2} = 6x - 6.$$

The roots of the equation

(21) $$3x^2 - 6x + 3 = 0$$

both are unity and the corresponding ordinate is $y = 3$. In the point whose coördinates are $x = 1$ and $y = 3$, the tangent of the curve is parallel to the x-axis. When $x = 1$, however, the second derivative becomes equal to zero; and the point in question does not present a maximum or minimum, but rather a point of inflexion.

ART. 6. **Minimum of intensity of heat.** *Let A and B be two point-sources of heat. It is required to find the point M*

on the straight line AB, which is at the lowest temperature, the intensity of the radiation of heat varying inversely as the square of the distance from the source of heat.

FIG. 66.

Let d represent the distance between the points A and B (Fig. 66), and x the distance from A of the point M on the straight line; then

(1) $$MA = x \text{ and } MB = d - x.$$

Let the intensities of the heat at unit distance from the sources of heat be denoted by α and β, respectively. The total intensity of heat ω at the point M is

(2) $$\omega = \frac{\alpha}{x^2} + \frac{\beta}{(d-x)^2},$$

and we wish to find for what values of x this expression is a minimum. We have

$$\frac{d\omega}{dx} = -\frac{2\alpha}{x^3} + \frac{2\beta}{(d-x)^3},$$

and the points we seek are determined by the equation

$$-\frac{2\alpha}{x^3} + \frac{2\beta}{(d-x)^3} = 0,$$

whence $$\frac{(d-x)^3}{x^3} = \frac{\beta}{\alpha}.$$

By extracting the cube root of both members we obtain

(3) $$\frac{d-x}{x} = \frac{\sqrt[3]{\beta}}{\sqrt[3]{\alpha}}.$$

The distances BM and AM have, therefore, the same ratio as the cube roots of the corresponding heat intensities. By solving the last equation we have, further,

(4) $$x = \frac{\alpha^{\frac{1}{3}} d}{\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}}.$$

In this case it is necessary to ascertain whether the value found corresponds to a maximum or a minimum. This is easily done by means of the second derivative,

$$(5) \qquad \frac{d^2\omega}{dx^2} = \frac{2 \cdot 3\,\alpha}{x^4} + \frac{2 \cdot 3\,\beta}{(d-x)^4},$$

which is positive for all values of x, inasmuch as the powers of x and $d - x$ are even, and α and β are essentially positive.

Art. 7. **The law of reflection.** *It is required to find a point so situated on the straight line GH* (Fig. 67) *that the sum of its distances from two fixed points A and B is a minimum.*

Let the perpendiculars AA' and BB' be designated by a and b, respectively; also let $A'B' = p$. P being any arbitrary point between A' and B', let its distance from A' be denoted by x, so that $PB' = p - x$. From the two right triangles $AA'P$ and $BB'P$, we have

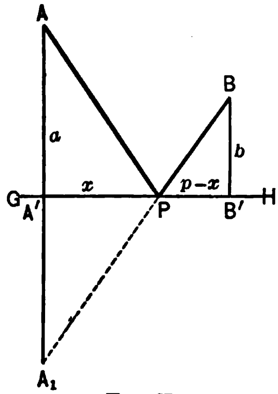

Fig. 67.

$$(1) \qquad AP = \sqrt{a^2 + x^2}, \qquad BP = \sqrt{b^2 + (p - x)^2},$$

the square roots being taken with the positive sign.

Since the sum of AP and BP is to be a minimum, we have to determine what value of x will make

$$(2) \qquad f(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (p - x)^2}$$

a minimum.

By differentiation we have

$$f'(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{(p - x)}{\sqrt{b^2 + (p - x)^2}};$$

and, since this must be equal to zero at a maximum or minimum, we have

$$(3) \qquad \frac{x}{\sqrt{a^2+x^2}} = \frac{(p-x)}{\sqrt{b^2+(p-x)^2}},$$

an equation which leads, by squaring, to a quadratic, whose roots are readily found to be

$$x = \frac{ap}{b-a} \text{ or } \frac{ap}{b+a}.$$

Of these values only the latter satisfies equation (3), the former having been brought in by squaring.

In order to ascertain whether a minimum actually exists we examine the second derivative,

$$(4) \qquad f''(x) = \frac{a^2}{(a^2+x^2)^{\frac{3}{2}}} + \frac{b^2}{(b^2+(p-x)^2)^{\frac{3}{2}}};$$

this is positive for all values of x, showing that the value of x found from (3) determines a minimum.

Equation (3) yields a simple geometric result. If we denote the angles APA' and BPB' by ϕ and ψ, we see from equation (3) that

$$\cos\phi = \cos\psi;$$

the minimum occurs at the point where the lines AP and BP make equal angles with the given straight line. If we conceive A to be a source of light and GH a reflecting surface, it is known that the ray of light AP will be reflected in the direction PB such that $\phi = \psi$; *i.e.* light which travels from A to B by reflection from GH takes a path which is a minimum.

It is easily seen that if the point A_1 be so taken in the straight line AA', and on the opposite side of GH that $A_1A' = A'A$, then the points A_1PB lie in a straight line.

ART. 8. **The law of refraction.** *Two points A and B lie on opposite sides of a straight line GH. If a point moves from A to B in the shortest time, and if its velocity is uniform but different on each side of the line, where does it cross the line?*

We drop perpendiculars from A and B to GH, and let $AA' = a$, $BB' = b$, and $A'B' = p$. Further, let P be any point on GH between A' and B', let x denote its distance $A'P$ from A', and $p - x$ its distance from B'. Let V_1 and V_2 be the velocities per second above and below GH respectively. Then the path APB is traversed in t seconds, where

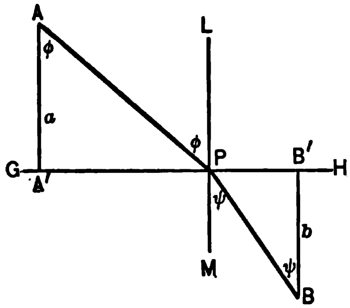

FIG. 68.

$$(1) \qquad t = \frac{AP}{V_1} + \frac{BP}{V_2}.$$

This is the expression which is to be examined for a minimum. In the triangles APA' and BPB' we see that

$$(2) \qquad AP = \sqrt{a^2 + x^2} \text{ and } BP = \sqrt{b^2 + (p-x)^2},$$

hence,

$$(3) \qquad t = \frac{\sqrt{a^2 + x^2}}{V_1} + \frac{\sqrt{b^2 + (p-x)^2}}{V_2},$$

Applying the criterion, we must have $\dfrac{dt}{dx} = 0$, or

$$(4) \qquad \frac{x}{V_1\sqrt{a^2 + x^2}} - \frac{(p-x)}{V_2\sqrt{b^2 + (p-x)^2}} = 0$$

This leads to an equation of the fourth degree when rationalized; but we can simplify matters by certain geo-

metric considerations. If at P we erect a perpendicular LM to GH, and denote the angles APL and BPM by ϕ and ψ, respectively, we have

$$\frac{x}{\sqrt{a^2+x^2}} = \sin\phi, \quad \frac{p-x}{\sqrt{b^2+(p-x)^2}} = \sin\psi.$$

Substituting these values, equation (4) becomes

$$\frac{\sin\phi}{V_1} = \frac{\sin\psi}{V_2},$$

or

$$(5) \qquad \frac{\sin\phi}{\sin\psi} = \frac{V_1}{V_2}.$$

There can be no minimum except for points between A' and B'; for if P be supposed to move beyond either of these, *both* parts of the path APB will constantly increase as P moves on, and therefore the time of traversing the path APB will also increase.

The actual solution of the equation resulting from equating the first derivative to zero, and the substitution of the value thus found in the second derivative, may often be avoided by proving from the geometrical conditions that a minimum or a maximum *must* exist. In the present case this is easily done. Taking any point P within the interval $A'B'$, points P_1 and P_2 beyond B' and A', respectively, exist such that AP_1B and AP_2B are both greater than APB. As the function is continuous, and does not become infinite between P_1 and P_2, there must be a minimum somewhere. We have already noticed that no minimum can occur except in $A'B'$, and in this interval there is only one point at which the necessary condition for a maximum or minimum is satisfied. The minimum must accordingly occur at this point, *i.e.* at the point P where the straight

lines AP and BP make with the perpendicular LM angles whose sines have the same ratio as the corresponding velocities. The actual determination of the position of P is of no special interest in this connection.

We now assume that GH separates two media of different composition. With each medium there is connected a constant, which is inversely proportional to the velocity of the propagation of light in the medium, and which is known as the **index of refraction.** (The standard of comparison, to which the index unity is assigned, is the vacuum.) If a ray of light passes from A to B, then the path taken by the ray is, in accordance with the Law of Refraction, such that the incident and refracted rays, *i.e.* AP and BP, make with the normal (LM) angles (ϕ and ψ), whose sines are inversely proportional to the indices of refraction; that is, inversely proportional to the velocities of the propagation of light in the media, so that the above equation represents the Law of Refraction. The ray is accordingly refracted so that the path AB is traversed in the shortest time.

EXERCISES XL

Examine the following for maxima and minima:

1. $y = \dfrac{x^2 - 7x + 6}{x - 10}$. *Ans.* max. $x = 4$; min. $x = 16$.

2. $y = x + \dfrac{a^2}{x}$ $(a > 0)$. *Ans.* max. $x = a$; min. $x = -a$.

3. $y = x + \sqrt{1 - x}$. *Ans.* max. $x = \frac{3}{4}$.

4. $y = x + \dfrac{1}{x}$. *Ans.* min. $x = +1$; max. $x = -1$.

5. $y = x^x$. *Ans.* max. $x = \dfrac{1}{e}$.

6. $y = x^{\frac{1}{x}}$. *Ans.* max. $x = e$.

7. $y = x^2 + 2px + q$. *Ans.* min. $x = -\dfrac{p}{q}$.

8. $y=(x-1)(x-2)(x-3)$. *Ans.* max. $x=2-\frac{1}{\sqrt{3}}$; min. $x=2+\frac{1}{\sqrt{3}}$.

9. $y=\frac{\log x}{x}$. *Ans.* max. $x=e$.

10. $y=x^p(a-x)^q(a>0;\ p, q, +\text{integers})$. *Ans.* $\begin{cases}\text{min. } x=0 \text{ if } p \text{ even.}\\ \text{min. } x=0 \text{ if } q \text{ even.}\\ \text{max. } x=\frac{pa}{p+q}.\\ \text{max. } x=a-\frac{aq}{p+q}.\end{cases}$

11. $y=\frac{x}{1+x^2}$. *Ans.* $\begin{cases}\text{max. } x=1.\\ \text{min. } x=-1.\end{cases}$

12. $y=\frac{(x+3)^3}{(x+2)^2}$. *Ans.* min. $x=0$.

13. $y=\frac{1-x+x^2}{1+x-x^2}$. *Ans.* min. $x=\frac{1}{2}$.

14. $y=\sin x(1+\cos x)$. *Ans.* $x=\frac{\pi}{3}$.

15. What fraction exceeds its square by the greatest number possible? *Ans.* $\frac{1}{2}$.

16. What rectangle of given perimeter has the largest area? *Ans.* The square.

17. Divide 10 into two such parts that the product of their cubes shall be as great as possible. *Ans.* 5, 5.

18. Given a cylindrical tree trunk of diameter D and of sufficient length; to cut from it a rectangular beam which shall have the greatest strength, given (from the theory of strains), that the resistance of a rectangular beam is directly proportional to bh^2, where b and h are the dimensions of the section of the beam, the pressure being applied perpendicularly to the side of breadth b. *Ans.* $h=\frac{D\sqrt{6}}{3}$; $b=\frac{D\sqrt{3}}{3}$.

19. A submarine telegraphic cable consists of a central circular part called the core, surrounded by a concentric circular part called the covering. If x denote the ratio of the radius of the core to that of the covering, it is known that the speed of signaling varies as $x^2\log\frac{1}{x}$. Show that the greatest speed is attained when $x=\frac{1}{\sqrt{e}}$.

20. From the corners of a given rectangular piece of tin (dimensions a and b) square pieces are to be cut, so that the pan formed by turning up the sides so produced shall have as great a volume as possible.

Ans. Side of squares cut out $= \dfrac{a + b - \sqrt{a^2 - ab + b^2}}{3}$.

21. What rectangle of given area has the smallest perimeter?

Ans. The square.

22. In a horizontal plane the distance d between two points A and B is known. Given, that the intensity of light varies directly as the sine of the angle of incidence; and, inversely, as the square of the distance, to find at what height h, perpendicularly above A, an incandescent point must be situated in order that the intensity of light from it at the point B may be at a maximum.

Ans. $h = \dfrac{d\sqrt{2}}{2}$.

23. An open bin, with square base and vertical sides, is to hold a given volume of wheat. What must be its inner dimensions in order that as little material as possible may be needed to construct it, the thickness of the material being disregarded?

Ans. The depth must be half the width.

24. An object, AB, of length l is a perpendicular to a given plane, and the lower end of AB is at the distance d from the plane. At what distance x from the point where AB produced meets the plane must an observer in the plane stand in order that the object may appear the largest?

Ans. $x = \sqrt{d(d + l)}$.

25. To inscribe the largest possible rectangle in a given triangle.

Ans. The altitude of the rectangle must be half the altitude of the triangle.

26. A person in a boat three miles from the nearest point P of a straight beach wishes to reach in the shortest time a place on the shore five miles from the point P; if he can run five miles per hour, but row only four miles per hour, at what place must he land?

Ans. One mile from the point to be reached.

27. A body is projected upwards at angle α with velocity c; to what height will it attain, disregarding the resistance of the air?

HINT. We know from physics that under the given conditions the body will attain, in x seconds, the height

$$cx \sin \alpha - \frac{gx^2}{2},$$

g being the constant of gravity. This is to be a maximum. *Ans.* $\dfrac{c^2 \sin^2 \alpha}{2g}$.

28. A perpendicular lamp post is to be erected at a given horizontal distance d from a statue. What must be the height of the lamp above the head of the statue in order that the top of the head may be most strongly illuminated?

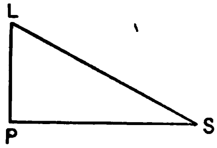

FIG. 69.

HINT. According to physics, the intensity of the illumination is inversely proportional to the square of the distance of the light from the illuminated point, and directly proportional to the sine of the angle at which the rays of light strike the object. We have then, that $\dfrac{\sin LSP}{(SL)^2}$ has to be a maximum, or, if $LP = x$, $PS = d$, $\dfrac{x}{(d^2 + x^2)^{\frac{3}{2}}}$ is to be a maximum. We may simplify this by squaring and putting $\dfrac{1}{d^2 + x^2} = z$ when the expression to be a maximum becomes

$$z^2 - d^2z^3.$$

Ans. $x = \dfrac{d}{2}\sqrt{2}$.

29. At what height on a perpendicular wall must letters of a given height, h, be placed in order to appear the largest to a spectator at a given distance d?

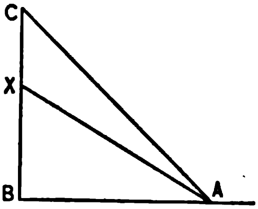

FIG. 70.

HINT. Let $AB = d$, $CX = h$, $BX = x$. Angle CAX is to be maximum.

We have

$$\cot CAX = \cot(CAB - XAB) = \frac{d^2 + x^2 + hx}{hd}.$$

This has to be minimum in order that angle may be maximum.

Ans. $x = -\dfrac{h}{2}$, *i.e.* the middle of letters must be on a horizontal line with eye.

30. What is the minimum of material (disregarding the thickness) needed to make a right cylindrical vessel, open at the top, of given volume V^2? *Ans.* $3\sqrt[3]{\pi V^2}$.

31. What is the minimum of material needed in the previous problem if the thickness, c, be taken into account?

Ans. $3\,c\sqrt[3]{\pi V^2} + 3\,c^2\sqrt[3]{\pi^2 V} + \pi c^3$.

ART. 9. **Estimation of errors.** It is not often possible or practicable to measure a given quantity *directly*. Usually

one or more quantities are measured which stand in a known relationship to the required one. Thus, the equivalent weight of sodium is not determined *directly* by preparing sodium chloride out of weighed amounts of sodium and chlorine, or by decomposing a known weight of sodium chloride into its components, but *indirectly* by finding what amount of silver can replace the sodium, and by calculating the required equivalent weight on the assumption that the composition of silver chloride is known; likewise, in the measurement of resistance by means of a Wheatstone Bridge the required quantity is not found directly but indirectly by calculation from the relationships of the arms of the Bridge; likewise, the reading of a galvanometer deflection does not give at once the strength of an electric current, but this is found only after the trigonometric tangent of the angle of deflection is known, etc., etc.

In all such cases as these, the methods of the present chapter are of great assistance in determining the best mode of conducting the experiments and of criticising the results. Let y be a required quantity to be determined from another quantity x directly measurable, and known to be connected with y by the relation

$$y = f(x).$$

Mathematically, the quantity y is hereby fixed as soon as x is known; but under our assumption x is to be determined experimentally, and accordingly can never be known accurately, but only approximately.

Each observed value of x will differ by some error Δx from the true value. This error Δx is of course not known, except that it lies between certain limits. Nevertheless, it has an influence upon the calculated value of y, causing the

latter to differ from the true value by an error which we call Δy. The error in the final result accordingly amounts to

$$(1) \qquad \Delta y = f(x+\Delta x) - f(x),$$

where x denotes the observed value and $x + \Delta x$ the true value.

By Taylor's Theorem,

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots.$$

Δx must certainly be a very small quantity, or the measurement itself is comparatively worthless. Hence, in the expansion into a Taylor's Series, we may omit all terms after the second, obtaining as a close approximation,

$$(2) \qquad \Delta y = f'(x)\Delta x,$$

and as the *relative error*,*

$$(3) \qquad \frac{\Delta y}{y} = \frac{f'(x)}{f(x)}\Delta x.$$

Equation (3) finds frequent applications in the critical revision of the results obtained according to different methods of experimentation.

EXAMPLES

1. In determining the equivalent weight of sodium (p. 384) it has been found that x parts of sodium chloride are precipitated by one part of dissolved silver (as silver chloride). Let A be the equivalent weight of silver, B that of chlorine (both of which are assumed to be

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* Evidently it is not the *absolute* but the *relative* error which determines the accuracy of a measurement. When we determine a weight to within 0.1 gram, the accuracy attained may in certain cases be very great, while in others it is entirely insufficient, since all depends upon whether 0.1 gram is an extremely small or a rather large fraction of the total weight. (See p. 357.) Multiplying the relative error by 100, we obtain the **percentage of error.**

known). It is further known that the required equivalent weight y of the sodium may be found by means of the equation

$$(y + B) : A = x : 1, \quad \text{or} \quad y = Ax - B.$$

The relative error is, according to equation (3),

$$\frac{\Delta y}{y} = \frac{f'(x)}{f(x)} \Delta x = \frac{A}{Ax - B} \Delta x,$$

or

$$\frac{\Delta y}{y} = \frac{A}{y} \Delta x = \frac{y + B}{y} \frac{\Delta x}{x}.$$

For the example in hand, $y = 23$, $B = 35.5$, approximately, and

$$\frac{y + B}{y} = \frac{23 + 35.5}{23} = 2.54;$$

so that an error of 1 per cent in the determination of x produces an error of about 2.5 per cent in y. It is a disadvantage to have B considerably larger than y. In the case of barium chloride where $y = \frac{137}{2}$ (equivalent weight of barium),

$$\frac{y + B}{y} = 1.52.$$

Hence, an equivalent weight determination of barium by precipitation of barium chloride with dissolved silver gives better results, other conditions being the same, than does that of sodium.

2. Let x denote the extent of a chemical reaction at the time t. The speed of reaction y (previously represented by k) may be found according to the considerations on p. 240 *et seq.* by means of the general formula

$$y = \frac{1}{t} \phi(x),$$

where $\phi(x)$ denotes a function depending upon the nature of the reaction in question; t as well as x must be measured in order to calculate y, but as a rule t can be determined so closely that there is no appreciable error, while x is affected by the error Δx. We therefore have

$$\Delta y = \frac{1}{t} \phi'(x) \Delta x.$$

When the number of observations of the same phenomenon is large (as in Chap. VII, p. 244), the values of Δx may be taken as equal, and the error of y calculated from each value of x may be put proportional to $\frac{\phi'(x)}{t}$, and the *reliability* of this value of y proportional to $\frac{t}{\phi'(x)}$. In

such cases it is not permissible to take the mean of the different values found for y, but each value of y is to be multiplied by the corresponding value of $\frac{t}{\phi'(x)}$ (called the **weight** of the observation), and divided by the sum of all the weights. We have accordingly the formula

$$y = \frac{y_1 \frac{t_1}{\phi'(x_1)} + y_2 \frac{t_2}{\phi'(x_2)} + \cdots}{\frac{t_1}{\phi'(x_1)} + \frac{t_2}{\phi'(x_2)} + \cdots},$$

in which each observation does not now exert equal (uncriticised) influence, but has an effect on the result determined according to its reliability. It is apparent that in this case it is a matter of indifference whether we carry out our computations with the relative error $\frac{\Delta y}{y}$, or more simply as we have done above, with the error Δy itself.

According to equation (3) the relative error is dependent upon Δx, the error of observation, and upon the quantity, $\frac{f'(x)}{f(x)}$. Both these factors are, therefore, to be made as small as possible. This is accomplished in the case of Δx by making our measurements as accurate as possible, and in the case of $\frac{f'(x)}{f(x)}$ by so arranging the experiment that this fraction becomes a minimum.

The last condition is fulfilled (p. 364) when the derivative

$$(4) \qquad \frac{d}{dx}\left[\frac{f'(x)}{f(x)}\right] = 0.$$

It is often impracticable to arrange the experiment so that this condition shall be fulfilled. How it is achieved, when practicable, will be illustrated in the following examples:

3. *Measurement of resistance by a Wheatstone Bridge.* The required resistance y is computed from the formula

$$(5) \qquad y = f(x) = w \frac{x}{l - x},$$

where w is the compensating resistance, l the length of the slide-wire, and x the position when a balance is secured. In this case

$$f'(x) = w \frac{l}{(l - x)^2}, \quad \frac{f'(x)}{f(x)} = \frac{l}{x(l - x)},$$

and finally,

$$\frac{d}{dx}\left[\frac{f'(x)}{f(x)}\right] = l \frac{2x - l}{x^2(l - x)^2}.$$

This expression becomes equal to zero when $x = \frac{l}{2}$. The error of balance (for instance, 0.1 mm.) has on this account the least influence on the end-result in the middle portions of the bridge-wire. It is therefore good practice to alter the compensating resistance so that in securing the balance only the middle portions of the wire may be used.

4. *Measurement of current strength with a tangent galvanometer.* The required current strength y is proportional to the tangent of the angle of deflection x; hence $y = f(x)$ is in this case,

$$y = C \tan x.$$

Now
$$f'(x) = \frac{C}{\cos^2 x}, \quad \frac{f'(x)}{f(x)} = \frac{1}{\sin x \cos x},$$

and
$$\frac{d}{dx}\left[\frac{f'(x)}{f(x)}\right] = \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}.$$

The last expression vanishes when $\sin x = \cos x$; that is, for an angle of 45°. The error of reading has therefore the smallest influence on the final result when in a given case the dimensions, turns of wire, etc., are so chosen that a deflection of 45° is obtained.

CHAPTER XII

DIFFERENTIATION AND INTEGRATION OF FUNCTIONS FOUND EMPIRICALLY

ART. 1. **Differentiation.** When by direct observation certain relationships between two variable quantities have been found, it is customary first of all to collect the results of the measurements into a table. We then endeavor according to circumstances either to find a mathematical expression (interpolation formula) that will enable us to compute with as good an approximation as possible the values of one quantity from those of the other, or we try to make the relationships found clearer by a graphic representation. While in many cases, moreover, the derivative of one of the quantities with respect to the other is of theoretic importance, its direct determination is of course impossible, because our instruments measure only with a certain degree of accuracy, and are therefore not able to follow by measurement beyond a certain point the value of the ratio of quantities which approach the limit zero. But if we are in possession of a sufficiently good interpolation formula, its differentiation will give the required result; * or if on the other hand we

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* Thus Horstmann (*Berichte der deutschen chemischen Gesellschaft*, Vol. 2, p. 137 [1869]), letting p denote the tension of dissociation of sal ammoniac, and t the temperature, made use of the interpolation formula

$$\log p = a + bA^t$$

(where a, b, and A are constants whose values may be taken from a table) to find the value of the derivative $\frac{dp}{dt}$.

have secured an accurate graphic representation, tangents drawn at the desired points of the curve determine the derivatives approximately (p. 117).*

Both methods have their shortcomings; the first one assumes that we are in possession of a good interpolation formula, which, however, we cannot obtain at all in many cases, and which almost always necessitates quite a little tedious computation; the second one requires unusual skill in drawing to attain results of much accuracy.

There is, however, a third method permitting of the determination of the approximate value of the derivative directly from a table of experimental results. Let $f(x)$ be the function in question, and let us suppose that we know its value for values of the variable differing by the same amount, as

$$(1) \qquad x,\ x \pm h,\ x \pm 2\,h,\ x \pm 3\,h,\ \cdots.$$

Such a problem arises, for example, when there is known the vapor tension p for a liquid at temperatures θ that differ among themselves by the same number of degrees (one degree, for instance), and it is required to find the derivative $\frac{dp}{d\theta}$ for a given tension $p = p_0$.

We give the formula at once, letting its proof come later.

$$(2) \qquad \frac{dp}{d\theta}=\frac{1}{h}\left\{\frac{\Delta_0+\Delta_{-1}}{2}-\frac{1}{6}\frac{\Delta''_{-1}+\Delta''_{-2}}{2}-\frac{1}{30}\frac{\Delta^{\text{iv}}_{-2}+\Delta^{\text{iv}}_{-3}\cdots}{2}\right\};$$

where the quantities Δ_0, Δ_{-1}, Δ''_{-1}, $\cdots$ have the following meaning. If we put

$$p_0=f(\theta),\ p_1 = f(\theta+h),\ p_2 = f(\theta+2\,h)\ \cdots,$$
$$p_{-1}=f(\theta-h),\ p_{-2}=f(\theta-2\,h),\ \cdots,$$

* This was the second way in which Horstmann (*Liebig's Annalen, Ergängzungsband*, 8, p. 125 [1871-1872]) found the derivative mentioned in the preceding footnote.

then
$$p_1 - p_0 = \Delta_0, \quad p_0 \ - p_{-1} = \Delta_{-1},$$
$$p_2 - p_1 = \Delta_1, \quad p_{-1} - p_{-2} = \Delta_{-2},$$
$$\text{etc.,} \qquad \text{etc.}$$

The quantities
$$\Delta_1, \Delta_0, \Delta_{-1}, \Delta_{-2}, \cdots$$
represent the differences of the successive values of the pressure; we call it the *first series of differences.* Furthermore, the quantities
$$\Delta_1 - \Delta_0 = \Delta'_0, \quad \Delta_0 \ - \Delta_{-1} = \Delta'_{-1},$$
$$\Delta_2 - \Delta_1 = \Delta'_1, \quad \Delta_{-1} - \Delta_{-2} = \Delta'_{-2},$$
$$\text{etc.,} \qquad \text{etc.;}$$
that is, the series
$$\Delta'_1, \Delta'_0, \Delta'_{-1}, \Delta'_{-2}, \cdots$$
represents the differences between the numbers of the first series of differences taken in order; it is called the *second series of differences.* Likewise,
$$\Delta''_1, \Delta''_0, \Delta''_{-1}, \Delta''_{-2}, \cdots$$
represent the differences between the successive numbers of the second series of differences, and we have the *third series of differences.* In like manner, we can proceed to form the *fourth, fifth,* and *higher series of differences,* but the series beyond the third are used very rarely.

We now proceed to illustrate the above formula by an example. It is required to find the value of $\frac{dp}{d\theta}$ at 100° C. from the values given by Wiebe * for the vapor tension p of water at the temperature θ. We find in his tables the following values of p and θ from 0°.5 to 0°.5:

* *Tafeln über die Spannkraft des Wasserdampfes,* Braunschweig, 1894.

θ	p	Δ	Δ'	Δ''
99.0	$(733.24)_{-2}$			
99.5	$(746.52)_{-1}$	$(13.28)_{-2}$	$(0.20)_{-2}$	
100.0	$(760.00)_0$	$(13.48)_{-1}$	$(0.21)_{-1}$	$(+0.01)_{-2}$
100.5	$(773.69)_{+1}$	$(13.69)_0$	$(0.20)_0$	$(-0.01)_{-1}$
101.0	$(787.58)_{+2}$	$(13.89)_1$		

We have attached the proper indices to the numbers in the above table, and by substitution in equation (1) we have

$$\frac{dp}{d\theta}=\frac{1}{0.5}\left\{\frac{13.48+13.69}{2}-\frac{1}{6}\frac{0.01-0.01}{2}\right\}=27.17;$$

that is, in the vicinity of 100° C., an increase or decrease of pressure amounting to 27.17 mm. of mercury corresponds to a rise or fall in temperature of one degree.

We now pass to the proof of our formula, employing the same quantities as in the example. Let

$$(3)\qquad p=f(\theta)=A+B\theta+C\theta^2+D\theta^3+E\theta^4+\cdots$$

be a series representing the pressure p as a function of θ. Its differentiation gives

$$(4)\qquad \frac{dp}{d\theta}=f'(\theta)=B+2\,C\theta+3\,D\theta^2+4\,E\theta^3+\cdots,$$

and it is now required to represent the coefficients of the series in terms of the numerical data of the table. These data give the values of p for the temperatures

$$\theta,\ \theta\pm h,\ \theta\pm 2\,h.$$

If in equation (3) we substitute $\theta+h$ and $\theta-h$ for θ, we obtain the following equations:

$$(5)\quad \begin{aligned} p_1&=f(\theta+h)\\ &=A+B(\theta+h)+C(\theta+h)^2+D(\theta+h)^3+E(\theta+h)^4\cdots,\\ p_{-1}&=f(\theta-h)\\ &=A+B(\theta-h)+C(\theta-h)^2+D(\theta-h)^3+E(\theta-h)^4\cdots.\end{aligned}$$

If we expand the binomials in the parentheses, and subtract equation (3), we have

$$\begin{aligned} p_1 - p_0 &= Bh + C(2\,\theta h + h^2) + D(3\,\theta^2 h + 3\,\theta h^2 + h^3) \\ &\quad + E(4\,\theta^3 h + 6\,\theta^2 h^2 + 4\,\theta h^3 + h^4) + \cdots, \\ p_0 - p_{-1} &= Bh + C(2\,\theta h - h^2) + D(3\,\theta^2 h - 3\,\theta h^2 + h^3) \\ &\quad + E(4\,\theta^3 h - 6\,\theta^2 h^2 + 4\,\theta h^3 - h^4) + \cdots. \end{aligned}$$

By adding these equations together and remembering that

$$p_1 - p_0 = \Delta_0 \text{ and } p_0 - p_{-1} = \Delta_{-1},$$

we have

$$\begin{aligned} \Delta_0 + \Delta_{-1} &= 2\,Bh + 2\,C \cdot 2\,\theta h + 2\,D \cdot (3\,\theta^2 h + h^3) \\ &\quad + 2\,E(4\,\theta^3 h + 4\,\theta h^3) + \cdots, \end{aligned}$$

or, after dividing by $2\,h$,

$$\begin{aligned} \frac{1}{h} \cdot \frac{\Delta_0 + \Delta_{-1}}{2} &= B + 2\,C\theta + 3\,D\theta^2 h + 4\,E\theta^3 + \cdots \\ &\quad + Dh^2 + 4\,E\theta h^2 + \cdots. \end{aligned}$$

The first line of the right member is the derivative as developed in equation (4), so that we may put

$$(6) \qquad \frac{dp}{d\theta} = \frac{1}{h}\left(\frac{\Delta_0 + \Delta_{-1}}{2}\right) - (Dh^2 + 4\,E\theta h^2 + \cdots).$$

If h be made very small, the terms into which h^2 enters as a factor will usually be so small that if they are omitted we still have an approximate equality, *i.e.* we have approximately

$$(7) \qquad \frac{dp}{d\theta} = \frac{1}{h}\left(\frac{\Delta + \Delta_{-1}}{2}\right).$$

We have thus derived the first term of our general formula, equation (2), by assuming that we should still have the desired degree of approximation if we neglect the series

$$Dh^2 + 4\,E\theta h^2 + \cdots$$

in equation (6).

If this should not be the case, a closer approximation can be obtained as follows:

In equation (3) we substitute $\theta + 2h$ or $\theta - 2h$ for θ, and thus get

$$p_2 = A + B(\theta + 2h) + C(\theta + 2h)^2 + D(\theta + 2h)^3 + E(\theta + 2h)^4 + \cdots,$$

$$p_{-2} = A + B(\theta - 2h) + C(\theta - 2h)^2 + D(\theta - 2h)^3 + E(\theta - 2h)^4 + \cdots.$$

Subtracting these equations from equations (5), and keeping in mind that

$$p_2 - p_1 = \Delta_1 \text{ and } p_{-1} - p_{-2} = \Delta_{-2},$$

we find, after some simple reductions, that

$$\Delta_1 = Bh + C(2\theta h + 3h^2) + D(3\theta^2 h + 9\theta h^2 + 7h^3) + E(4\theta^3 h + 18\theta^2 h^2 + 28\theta h^3 + 15h^4) + \cdots,$$

$$\Delta_{-2} = Bh + C(2\theta h - 3h^2) + D(3\theta^2 h - 9\theta h^2 + 7h^3) + E(4\theta^3 h - 18\theta^2 h^2 + 28\theta h^3 - 15h^4) + \cdots.$$

We have previously found (p. 393)

$$\Delta_0 = Bh + C(2\theta h + h^2) + D(3\theta^2 h + 3\theta h^2 + h^3) + E(4\theta^3 h + 6\theta^2 h^2 + 4\theta h^3 + h^4) + \cdots;$$

$$\Delta_{-1} = Bh + C(2\theta h - h^2) + D(3\theta^2 h - 3\theta h^2 + h^3) + E(4\theta^3 h - 6\theta^2 h^2 + 4\theta h^3 - h^4) + \cdots;$$

and subtract these four equations from one another; in accordance with the notation introduced on p. 391, we have

$$\Delta' = C \cdot 2h^2 + D(6\theta h^2 + 6h^3) + E(12\theta^2 h^2 + 24\theta h^3 + 14h^4) + \cdots;$$

$$\Delta'_{-1} = C \cdot 2h^2 + D(6\theta h^2) + E(12\theta^2 h^2 + 2h^4) + \cdots;$$

$$\Delta'_{-2} = C \cdot 2h^2 + D(6\theta h^2 - 6h^3) + E(12\theta^2 h^2 - 24\theta h^3 + 14h^4) + \cdots$$

From these still another subtraction gives us for the new differences $\Delta''_{-1} = \Delta' - \Delta'_{-1}$ and $\Delta''_{-2} = \Delta'_{-1} - \Delta'_{-2}$ the values

$$\Delta''_{-1} = D \cdot 6\,h^3 + E(24\,\theta h^3 + 12\,h^4) + \cdots;$$
$$\Delta''_{-2} = D \cdot 6\,h^3 + E(24\,\theta h^3 - 12\,h^4) + \cdots.$$

On adding these two expressions, we obtain

$$\Delta''_{-1} + \Delta''_{-2} = 2\,D \cdot 6\,h^3 + 2\,E \cdot 24\,\theta h^3 + \cdots;$$

or finally, after dividing by $2\,h$,

$$\begin{aligned}\frac{1}{h} \cdot \frac{\Delta''_{-1} + \Delta''_{-2}}{2} &= 6\,Dh^2 + 24\,E\theta h^2 + \cdots \\ &= 6\,(Dh^2 + 4\,E\theta h^2 + \cdots) + \cdots.\end{aligned}$$

But the right member of this equation is just the one whose value we have been seeking. If we neglect the remaining right-hand terms and substitute the approximate* value thus found in equation (6), we have

$$\text{(8)} \qquad \frac{dp}{d\theta} = \frac{1}{h}\left\{\frac{\Delta_0 + \Delta_{-1}}{2} - \frac{1}{6}\frac{\Delta''_{-1} + \Delta''_{-2}}{2}\right\}$$

as a better approximation to the required derivative.

A still more exact formula is the one given above,

$$\frac{dp}{d\theta} = \frac{1}{h}\left\{\frac{\Delta_0 + \Delta_{-1}}{2} - \frac{1}{6}\frac{\Delta''_{-1} + \Delta''_{-2}}{2} - \frac{1}{30}\frac{\Delta^{\text{iv}}_{-2} + \Delta^{\text{iv}}_{-3}}{2}\right\},$$

where Δ^{iv}_{-2} and Δ^{iv}_{-3} have meanings entirely analogous to those of the foregoing quantities, and the formula is proved in a similar way.

ART. **2. Integration.** Oftentimes it is necessary to calculate the value of the definite integral

$$\int_{x_0}^{x_n} y\,dx$$

* Approximate, because if the terms $F\theta^5 + G\theta^6 + \cdots$ were taken into consideration, we should have additional terms multiplied by h^4, h^6, $\cdots$.

from the data of a table. To this end we may either integrate a suitable interpolation formula which conforms sufficiently closely to the observed values, or we may plot the curve and determine the area representing the value of the integral. (See p. 253.)

If for the values of x,

$$x_0, \ x_1, \ x_2, \ x_3, \ \cdots, \ x_n,$$

sufficiently close together, there are known corresponding values of y,

$$y_0, \ y_1, \ y_2, \ y_3, \ \cdots, \ y_n,$$

the required integral θ may as a first approximation be put equal to

$$(1) \qquad \theta = (x_1 - x_0)\frac{y_0 + y_1}{2} + (x_2 - x_1)\frac{y_1 + y_2}{2} + \cdots + (x_n - x_{n-1})\frac{y_{n-1} + y_n}{2};$$

this formula gives the sum of the areas of the trapezoids formed by the axis of abscissas, any two neighboring y-coordinates and the lines connecting their extremities; evidently it should be used only when these connecting lines approximate sufficiently near to the curve.

Example. At the time-intervals $t_0, t_1, t_2, \cdots, t_n$, let the strength of an electrical current be found to have the values $C_0, C_1, C_2, \cdots, C_n$. If the current passed through a silver solution, the amount of silver precipitated is equal to the product of the equivalent weight of the metal by the quantity of electricity E (electro-chemically measured) which has passed through the circuit. But the theory of electricity shows that

$$E = \int_{t_0}^{t_n} C\,dt;$$

from the observed values we obtain accordingly the following approximation to the value of E:

$$E = (t_1 - t_0)\frac{C_0 + C_1}{2} + (t_2 - t_1)\frac{C_1 + C_2}{2} + \cdots + (t_n - t_{n-1})\frac{C_{n-1} + C_n}{2}.$$

A closer approximation is secured by passing a parabola through every three consecutive extremities of the ordinates. To find the parabola, for instance, which passes through the ends of the ordinates whose values are y_0, y_1, y_2, we consider the curve

$$y = f(x) = y_0 + a(x - x_0) + b(x - x_0)^2. \tag{2 a}$$

This curve passes through the point (x_0, y_0), as is seen by direct substitution of these values. It will pass through the points (x_1, y_1) and (x_2, y_2) also, if a and b are so determined that

$$y_1 = y_0 + a(x_1 - x_0) + b(x_1 - x_0)^2, \tag{2}$$

$$y_2 = y_0 + a(x_2 - x_0) + b(x_2 - x_0)^2. \tag{3}$$

By solving these equations we find the values of a and b (expressed in terms of known quantities), which must be used in equation (2 a), in order that the resulting curve may pass through the three given points.

The curve is recognized as a parabola, because by transformation of coördinates (see pp. 59 *et seq.*) its equation can be brought into the form $\eta = 2p\xi^2$. We accomplish this by putting

$$x = \xi + \alpha, \quad y = \eta + \beta,$$

and giving such values to α and β that in the new equation the constant term and the coefficient of ξ vanish. We observe also that the ξ- and η-axes are parallel to the x- and y-axes, so that the portion of a parabola passing through the points 1, 2, 3 belongs to a parabola whose axis is parallel to that of X. It can be proved that a parabola is fully determined by three of its points and the direction of its axis.

The solution of equations (2) and (3) gives

$$a = \frac{(y_1 - y_0)(x_2 - x_0)^2 - (y_2 - y_0)(x_1 - x_0)^2}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}, \tag{4}$$

$$b = \frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}, \tag{5}$$

and integrating y with respect to x between the limits x_0 and x_2, we have

$$\int_{x_0}^{x_2} y\,dx = y_0(x_2 - x_0) + \frac{a}{2}(x_2 - x_0)^2 + \frac{b}{3}(x_2 - x_0)^3; \tag{6}$$

where a and b have the values given in equations (4) and (5).

We may treat in a similar manner x_2, x_3, x_4, and the corresponding values y_2, y_3, y_4, respectively; then x_4, x_5, x_6, and y_4, y_5, y_6, and so on. If n be even, we obtain the values of a series of integrals terminating with

$$\int_{x_{n-2}}^{x_n} y\,dx,$$

and the value of the required integral is

$$\theta = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \cdots + \int_{x_{n-2}}^{x_n} y\,dx.$$

If n be odd, an additional pair of values of x and y may be determined by observation or by interpolation, or we can also in one case compute the area of the surface comprehended between two successive coördinates as a trapezoid.

If the distances between the ordinates are all the same, so that

$$x_1 - x_0 = x_2 - x_1 = \cdots x_n - x_{n-1} = h,$$

the above equation may be simplified, and assumes the form

$$\theta = \frac{h}{3}[y_0 + y_n + y(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})],$$

an expression known as **Simpson's Formula** (of course n is still an even number). By planning the observations (p. 396) so that this formula can be used, much labor of computation may be avoided. We accomplish this in the above example, for instance, by reading the current strength at equal time intervals.

Numerical Example. Suppose we have given the corresponding values

$$x_0 = 1.000, \qquad y_0 = 0.5000,$$
$$x_1 = 1.500, \qquad y_1 = 0.3077,$$
$$x_2 = 2.000, \qquad y_2 = 0.2000,$$

and it is required to find the value of

$$\theta = \int_{x_0}^{x_2} y\,dx.$$

Formula (1) gives us, as an approximate value of I,

$$\theta = 0.5\,\frac{0.5000 + 0.3077}{2} + 0.5\,\frac{0.3077 + 0.2000}{2} = 0.32885.$$

Porter

If we use formula (6) instead, we may expect a closer approximation; we find, in fact,

$$\begin{aligned} y_0(x_2 - x_1) &= +\,0.5000 \\ \frac{a}{2}(x_2 - x_1) &= -\,0.2346 \\ \frac{b}{2}(x_2 - x_0)^2 &= +\,0.0564 \\ \hline \theta &= 0.3218 \end{aligned}$$

Since, in the example,

$$x_2 - x_1 = x_1 - x_0,$$

we can apply the simpler formula (7) more conveniently, and thus obtain

$$\theta = \frac{0.5}{3}[0.5000 + 4 \times 0.3077 + 0.2000] = 0.3218,$$

a value which, from the nature of the case, must be exactly the same as that obtained from equation (6).

In order to test the closeness of our approximation, the values of y in this example were not determined by observation, but from the equation

$$y = f(x) = \frac{1}{1 + x^2};$$

so that knowing exactly the relation between x and y, we could determine accurately for comparison the value of the required integral. It is obtained by integrating, with the result

$$\theta = \int_1^2 \frac{dx}{1 + x^2} = \text{arc tan } 2 - \text{arc tan } 1 = 0.321751.$$

We see, therefore, that equations (6) and (7) give quite close approximations.

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